Strongly Aperiodic Subshifts of Finite Type on Hyperbolic Groups

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joint with
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Theorem

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(We'll have to first describe our terms!)

- (word/Gromov) hyperbolic groups,
- and their ends;
- subshifts of finite type on a group,
- and the strongly aperiodic ones

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Quick examples: ...

Subshifts of finite type

On a group G, with an alphabet A, a subshift is a G invariant, closed subset of A^G (with the natural action $(x \cdot g)(h) = x(gh)$; a subshift of finite type is specified by a finite collection of finite forbidden (or allowed) patterns.

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(We have similar definitions for matching rule tiling spaces (in the plane, $\mathbb{H}^n$, etc), but there are some subtle differences.)

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(Note that each of these examples contain a non-trivial product of infinite groups.)

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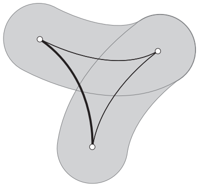
and Jeandel showed that finitely generated (*) groups with an undecidable word problem cannot either.

These are the only known obstructions and we naturally ask:

Question: *Does there exist a one ended finitely presented group with decidable word problem that does* not *admit a strongly aperiodic SFT?*

Hyperbolic groups

A group is **hyperbolic** (aka word-hyp. or Gromov hyp.) iff for some $\delta \geq 0$, every triangle is δ-**slim**:



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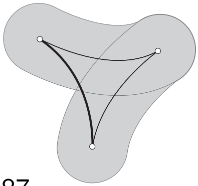

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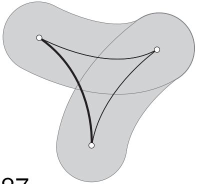

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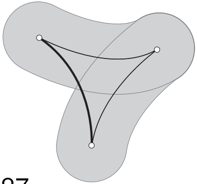

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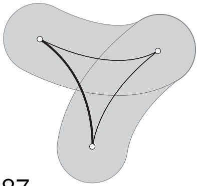

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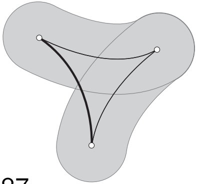

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Notably no hyperbolic group has a subgroup that is the non-trivial product of infinite groups (particularly $\mathbb{Z}^2$).

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- are in some sense ‘generic’ among finitely presented groups;
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For us, the key properties are that hyperbolic groups

- have well defined "horospherical shellings" and
- are "shortlex geodesic".
- And in particular, admit weakly aperiodic SFTs, we might call "shortlex shellings" (Gromov '87, Coornaert and Papadopoulous '91,'93.)

Sketch of the proof

Theorem. *A hyperbolic group admits a strongly aperiodic subshift of finite type if and only if it has at most one end.*

The primary result is

Prop. *A one-ended hyperbolic group admits an SFT such that no element is stabilized by* $\mathbb{Z}$.

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and every group with finitely many torsion elements up to conjugacy (eg hyp gps) admits an SFT s.t. no elt. has finite stabilizer... which can be combined with the proposition to produce a strongly aperiodic SFT.

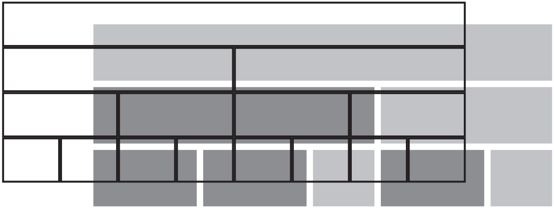
Incommeasurability

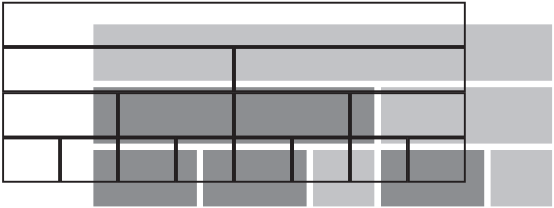
Consider the “orbit tiling” in $\mathbb{H}^2$ corresponding to the symbolic substitutions

$$0 \to 00$$

and

$$\mathtt{a} \to \mathtt{aab}, \mathtt{b} \to \mathtt{ab}$$





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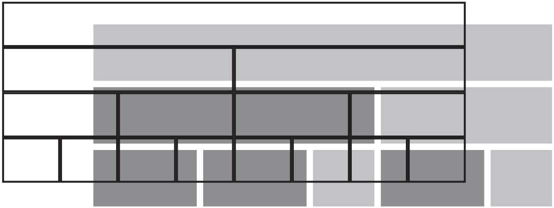

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In fact, the results of GS (strongly aperiodic tiles in $\mathbb{H}^2$) and Aubrun-Kari (weakly aperiodic tiles in the Baumslag-Solitar groups) can be understood as arising from the orbit tilings corresponding to the pair $0 \to 00$, $0 \to 000$.

The construction

The construction here is similar, though much more technical! Let λ be the asympotic exponential growth rate of the group; we associate a measure μ with the shortlex states of the group so that

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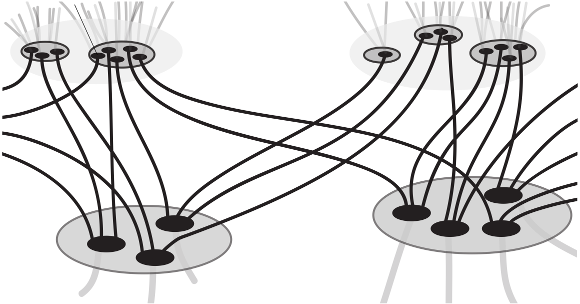
$$\sum_{P(b)=a} \mu(b) = \lambda \mu(a)$$

(This is essentially, but not quite, the leading left Perron-Frobenius eigenvector — the transition matrix need not be irreducible.)

We choose any $q \in \{2, 3\}$ for which $\log_q \lambda \notin \mathbb{Q}$.

Populated shellings

We will overlay a structure onto the shortlex shelling SFT:



In a **populated shelling**, each "village" $g \in G$ has $\lfloor \mu(g) \rfloor$ or $\lceil \mu(g) \rceil$ "villagers"; there is some global radius R, and some sequence $(\delta_i) \in \{\lfloor \log_q \lambda \rfloor, \lceil \log_q \lambda \rceil\}^{\mathbb{Z}}$, so that the villagers in layer i are in δ_i-to-one association with the villagers in layer $i+1$, within distance R.

This is a (possibly empty) SFT, as we can locally check the validity of an alleged populated shelling.

Populated shellings are strongly aperiodic

If they exist, populated shellings are strongly aperiodic, much as in the orbit tiling case. Consider any stabilizer S of a populated shelling x. S must preserve the underlying shortlex shelling, and so must act on the sequence (δ_i).

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On the other hand, a non-trivial S cannot fix any single layer, for the orbit of a point under such an S defines a quasigeodesic, in a shortlex layer, but quasigeodesics must remain close to geodesics, which diverge arbitrarily far from such a layer.

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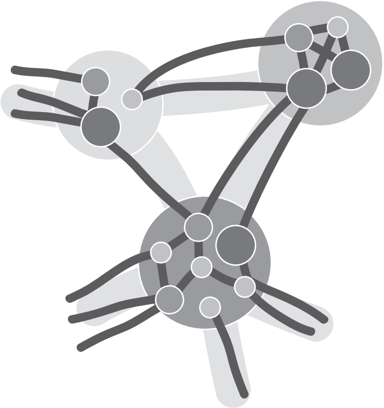
We construct "divergence graphs" on the shortlex layers which behave nicely under the shelling map.



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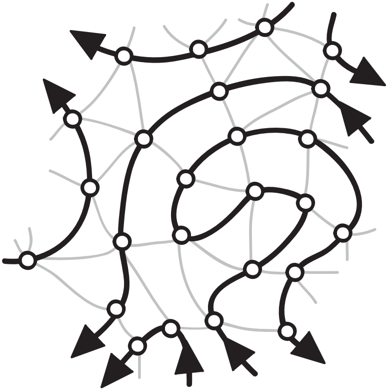
We construct a kind of flow on these graphs, which allows us to distribute errors in a controlled manner.



Populated shellings exist

We construct R very carefully to ensure that there are enough villagers in each layer to apply the Hall marriage theorem, allowing us to associate villagers with their children.