Multidimensional SFTs with countably many configurations

Ilkka Törmä

University of Turku, Finland

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Université de Lyon
Multidimensional symbolic dynamics

- Dimension $d$, finite alphabet $\Sigma$ with discrete topology
- Configuration: grid of symbols $x \in \Sigma^\mathbb{Z}^d$
- Full shift: $\Sigma^\mathbb{Z}^d$ with product topology, $\mathbb{Z}^d$ acts continuously by translations
- Shift space: closed translation-invariant subset of $\Sigma^\mathbb{Z}^d$; defined by forbidden finite patterns
- Shift of finite type (SFT): defined by finitely many forbidden patterns
- Sofic shift: symbol-to-symbol image of an SFT
Example (2D Even Shift)

Alphabet $\Sigma = \{\square, \bullet, \circ, \ast, \star, \odor, \dagger, \ddagger, \star, \heartsuit\}$, forbid $2 \times 1$ or $1 \times 2$ patterns where sides don’t match.
Multidimensional symbolic dynamics

Example (2D Even Shift)

Alphabet $\Sigma = \{\square, \circ, \ast, \bullet, \oplus, \ominus, \equiv, \vdash, \dashv\}$, forbid $2 \times 1$ or $1 \times 2$ patterns where sides don’t match.

Forget gray lines, obtain sofic shift of even-sized connected components of $\square$-symbols.
Countability

- Most well-known shift spaces contain uncountably many configurations
- Full shifts, mixing SFTs and sofic shifts, Robinson tilings, self-simulating tilings, substitutive shifts, Toeplitz shifts, Sturmian shifts...
- We consider *countable* shift spaces, in particular countable SFTs
Countability

Example (Sunny side up shift)

Forbid all finite patterns that contain more than one 1:

This is a countable 2D sofic shift.
Countability

Example (2D grid shift)

Forbid all $2 \times 2$ patterns not in this figure:

Countable 2D SFT with infinitely many periodic orbits.
Countable 2D SFT with infinitely many periodic orbits.
Countable Shift Spaces

Ilkka Törmä

Introduction

Countability

Structure of countable SFTs

Hierarchical structures

Growth rates

Problems

Countability

Example (2D grid shift)

Countable 2D SFT with infinitely many periodic orbits.
Example (2D grid shift)

Countable 2D SFT with infinitely many periodic orbits.
Countability

Example (2D grid shift)

Countable 2D SFT with infinitely many periodic orbits.
Some topology

- Topology of $\Sigma^\mathbb{Z}_d$ is generated by \textit{cylinder sets}
  
  $[P] = \{ x \in \Sigma^\mathbb{Z}_d : x|_D = P \}$ for finite patterns $P \in \Sigma^D$

- A configuration $x \in X$ of shift space is isolated iff there is a finite pattern $P \in \Sigma^D$ st. $X \cap [P] = \{x\}$

- We say $P$ isolates $x$ in $X$
Example: Computability of isolated points

Proposition

If $X \subset \Sigma^{\mathbb{Z}^d}$ is an SFT, then each of its isolated configurations is computable.

Proof.

Let $x \in X$ be isolated by a finite pattern $P$. For $n \in \mathbb{N}$, consider all possible ways of extending $P$ to $[-n, n]^d$ that avoid the forbidden patterns of $X$. Since $P$ isolates $x$ in $X$, the symbol at each coordinate $\vec{v} \in \mathbb{Z}^d$ is eventually fixed to $x_{\vec{v}}$. This gives an algorithm to compute $x_{\vec{v}}$ from $\vec{v}$. \qed
Example: Aperiodic implies uncountable

Proposition

A countable shift space $X$ contains a fully periodic configuration

Proof.

Let $Y = \overline{O(y)} \subseteq X$ be a minimal subsystem. We claim that $Y$ is finite. If $Y$ is infinite, then the set of limit points of $O(y)$ is all of $Y$, so $Y$ is perfect. A nonempty perfect complete metric space is uncountable, a contradiction. \qed
Example: Aperiodic implies uncountable

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Theorem (Ballier, Durand, Jeandel 2008)

*An infinite countable 2D SFT contains a configuration with exactly one direction of periodicity*
More topology

- **Cantor-Bendixson derivative** of $X$ is $X' = \{x \in X : x \text{ not isolated in } X\}$
- Transfinite iteration: $X^{(0)} = X$, $X^{(\alpha+1)} = (X^{(\alpha)})'$, $X^{(\lambda)} = \bigcap_{\beta < \lambda} X^{(\beta)}$ for limit ordinal $\lambda$
- CB-rank $r(X)$ is lowest ordinal $\lambda$ with $X^{(\lambda+1)} = X^{(\lambda)}$
- A shift space $X$ is countable iff $X^{(r(X))} = \emptyset$
- Rank $\rho(x)$ of $x \in X$ is lowest ordinal $\lambda$ with $x \notin X^{(\lambda)}$
Even more topology

- Subpattern order: $x \prec y$ if $\mathcal{O}(x) \subsetneq \mathcal{O}(y)$ (y contains all finite patterns of x)
- Subpattern poset $P(X)$: orbit closures $\mathcal{O}(x)$ for $x \in X$, ordered by inclusion, as abstract poset
- If $x \prec y$, then $\rho(x) \geq \rho(y)$ (if the ranks exist)
- $\prec$-minimal element $\iff$ minimal orbit closure
- $\prec$-maximal elements exist in all shift spaces
Subpattern order

Example of ranks and order $\prec$ in a countable SFT

```
1 1 1
...
2 2 1
3 3 3
```
## Structure of countable SFTs

<table>
<thead>
<tr>
<th>Theorem (Ballier, Jeandel 2013)</th>
</tr>
</thead>
<tbody>
<tr>
<td>In a rank-4 countable 2D SFT, configurations of rank</td>
</tr>
<tr>
<td>- 4 are doubly periodic</td>
</tr>
<tr>
<td>- 3 are periodic</td>
</tr>
<tr>
<td>- 2 are periodic or ‘star-shaped’</td>
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<tr>
<td>- 1 are computable</td>
</tr>
</tbody>
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<th>Theorem (T. 2015)</th>
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<td>There is a rank-5 countable SFTs with an uncomputable configuration</td>
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</table>

Proved by embedding computation into the SFT
How to embed computation into countable SFT?

- Infinite tape of Turing machine $\Rightarrow$ uncountability
- *Counter machine*: finite state set, fixed finite number of counters with values in $\mathbb{N}$
- Can increment/decrement any counter and check if it holds the value 0
- Counter machines are computationally universal
Example (Simulation of computation)

Simulating a counter machine in a countable SFT:
Theorem (Ballier, Durand, Jeandel 2008)

If $X$ is a countable shift space, then $P(X)$ contains no infinite ascending chains

Proof.

The subshift $Y = X^{(r(X))} \subseteq X$ is a perfect space. Since $X$ is countable, $Y = \emptyset$, and thus every configuration $x \in X$ has a rank $\rho(x)$. Recall that $x \prec y$ implies $\rho(x) > \rho(y)$. An infinite ascending chain in $P(X)$ would give an infinite descending chain of ordinals, which is impossible.
Descending chains

**Theorem (Salo, T. 2013)**

*There exists a countable 2D SFT $X$ such that $P(X)$ contains an infinite descending chain*

Construction involves a hierarchical structure different from the one present in Robinson’s tiles, substitutions etc.
Descending chains

Embedding a chain $x_1 \succ x_2 \succ x_3 \succ x_4 \succ \cdots$

$x_k$:

width $k$

width $k + 1$
Hyperarithmetic posets

- A set $Q \subset \mathbb{N}$ is **hyperarithmetic**, if it is the unique infinite branch of a computable directed tree on $\mathbb{N}$
- A **hyperarithmetic poset** is a poset $(Q, \leq)$ such that $Q \subset \mathbb{N}$ and $\leq \subset Q \times Q$ are hyperarithmetic
- $P(X)$ is hyperarithmetic for every countable SFT $X$

**Theorem (T. 2015)**

For each hyperarithmetic poset $(Q, \leq)$ with no infinite ascending chain, there exists countable 2D SFT $X$ and order-embedding $\phi : (Q, \leq) \to P(X)$ such that all elements of $P(X) \setminus \phi(Q)$ have height $\leq 3$
Pattern complexity of shift space $X$ is the number of distinct size-$n$ blocks: $P_n(X) = \# \{ x | [1,n]^d : x \in X \}$

Topological entropy: $h(X) = \lim_n \frac{\log P_n(X)}{n^d}$

Countable shift spaces have zero entropy (equivalently, $P_n(X)$ grows sub-exponentially in $n^d$)

Pattern complexity of an SFT is upper semi-computable
Growth rates of countable SFTs

**Theorem (T. 2017, unpublished)**

*Given a sub-exponential upper semi-computable function* $f : \mathbb{N} \to \mathbb{N}$, *there exists a countable 2D SFT* $X$ *with* $P_n(X) > f(n^2)$ *for all* $n \geq 1$.

Something stronger is probably also true.
Open problems

- High-level structure of subpattern posets has been characterized; what about the rest?
- What are the possible CB-ranks of countable SFTs?
  - [Ballier, Jeandel 2013] It cannot be $\lambda + n$ for a limit ordinal $\lambda$ and $n \in \{0, 1, 2\}$
  - [T. 2015] It can be $\lambda + 4$ for any computable ordinal $\lambda$
  - Is $\lambda + 3$ possible?
- Do all countable sofic shifts have countable SFT covers?