Multidimensional SFTs with countably many configurations

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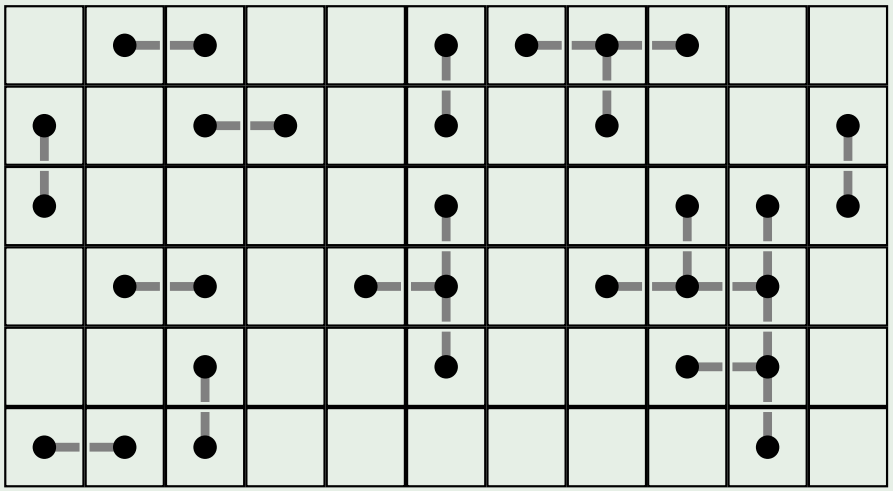
Multidimensional symbolic dynamics

- Dimension d, finite alphabet Σ with discrete topology
- Configuration: grid of symbols $x \in \Sigma^{\mathbb{Z}^d}$
- Full shift: $\Sigma^{\mathbb{Z}^d}$ with product topology, $\mathbb{Z}^d$ acts continuously by translations
- Shift space: closed translation-invariant subset of $\Sigma^{\mathbb{Z}^d}$; defined by forbidden finite patterns
- Shift of finite type (SFT): defined by finitely many forbidden patterns
- Sofic shift: symbol-to-symbol image of an SFT

Multidimensional symbolic dynamics

Example (2D Even Shift)

Alphabet $\Sigma = \{\square, \boxdot, \boxdot, \boxdot, \boxdot, \boxplus, \boxplus, \boxplus, \boxplus\}$, forbid 2×1 or 1×2 patterns where sides don't match.



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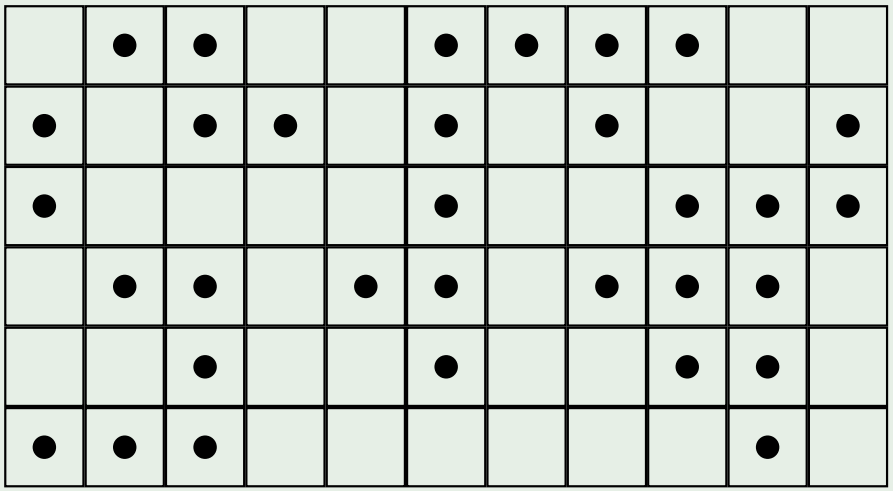


Forget gray lines, obtain sofic shift of even-sized connected components of $\boxdot$-symbols.

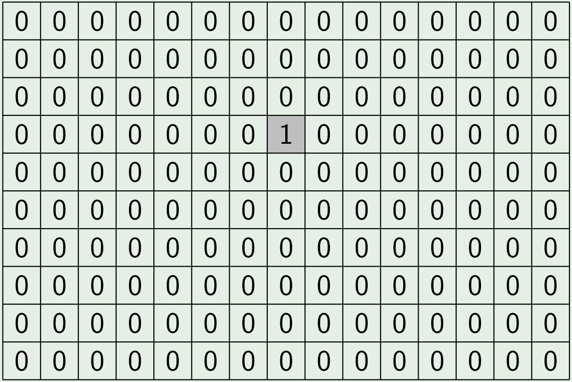
Countability

- Most well-known shift spaces contain uncountably many configurations
- Full shifts, mixing SFTs and sofic shifts, Robinson tilings, self-simulating tilings, substitutive shifts, Toeplitz shifts, Sturmian shifts. . .
- We consider *countable* shift spaces, in particular countable SFTs

Countability

Example (Sunny side up shift)

Forbid all finite patterns that contain more than one 1:



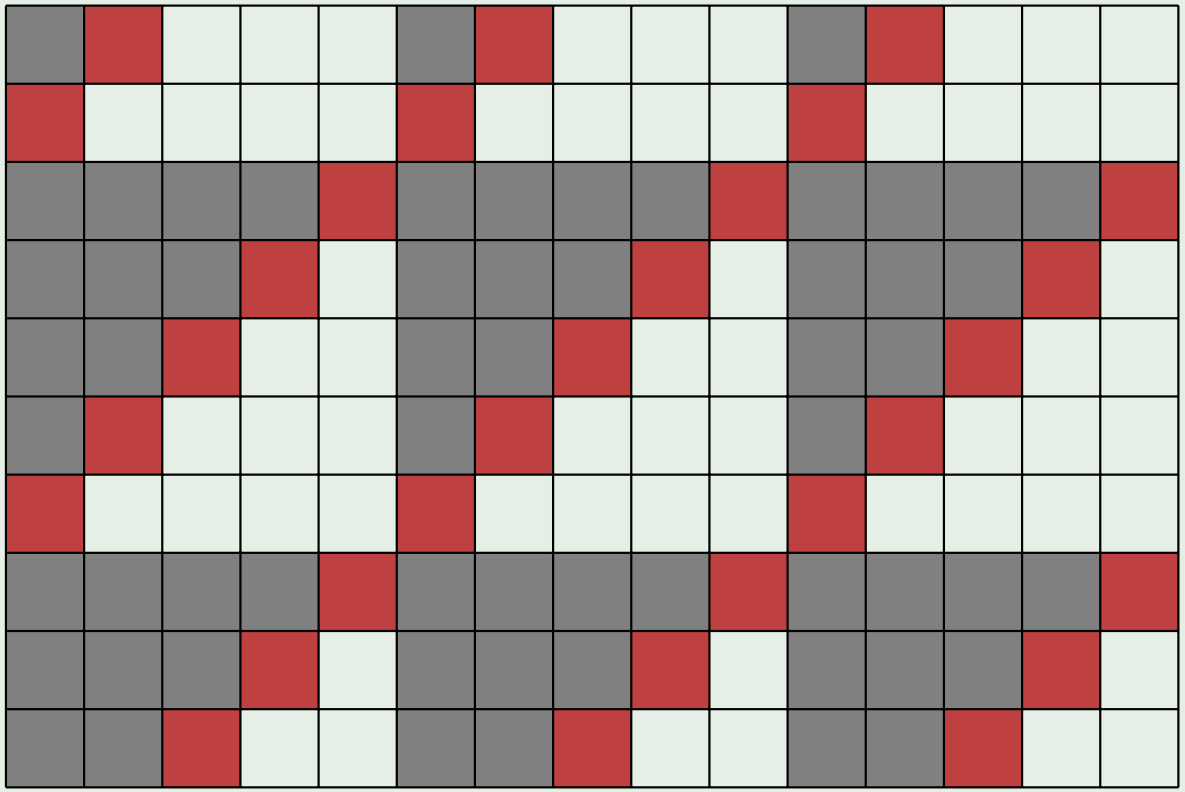
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

This is a countable 2D sofic shift.

Countability

Example (2D grid shift)

Forbid all 2×2 patterns not in this figure:



Countable 2D SFT with infinitely many periodic orbits.

Countability

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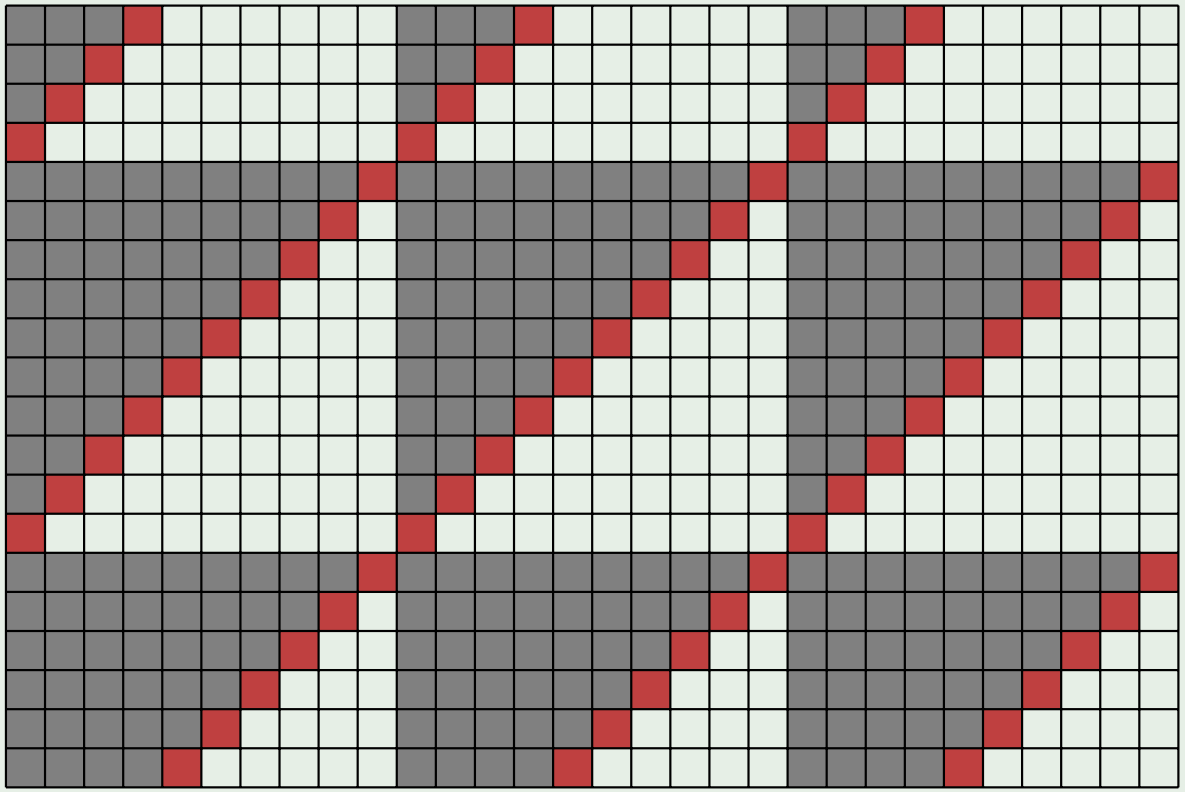

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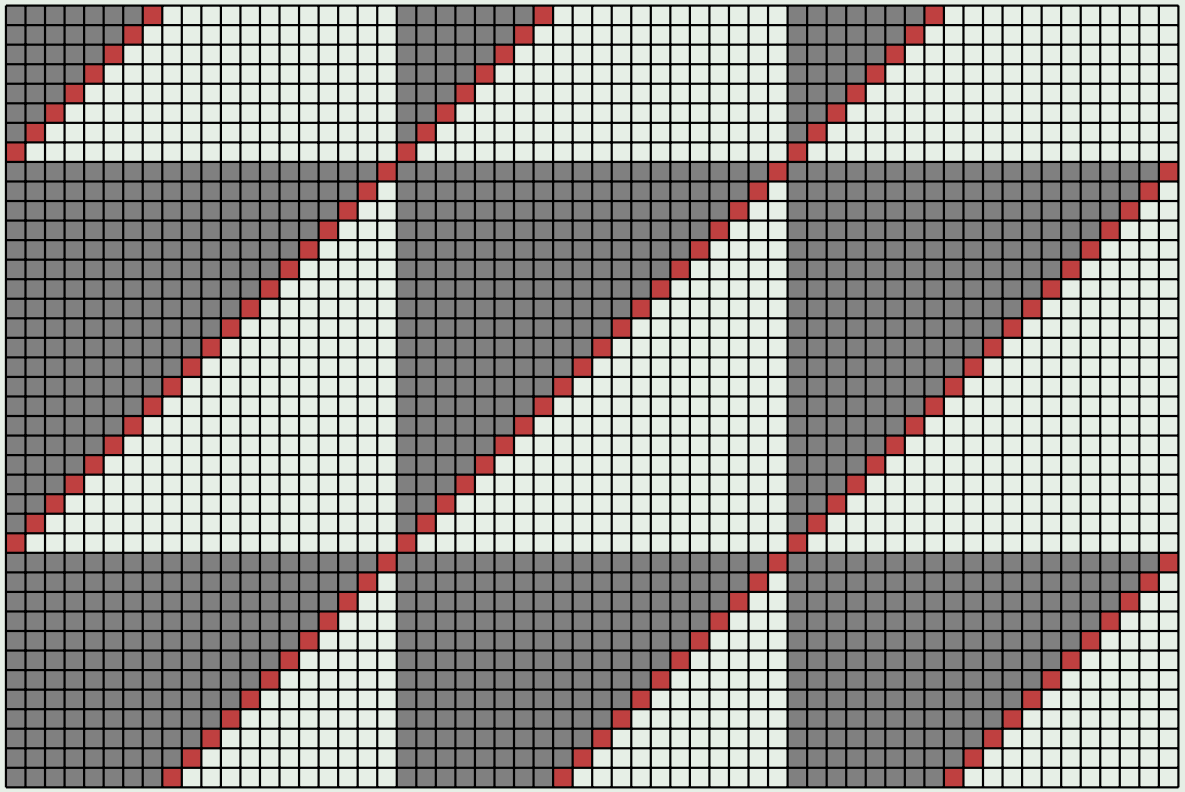

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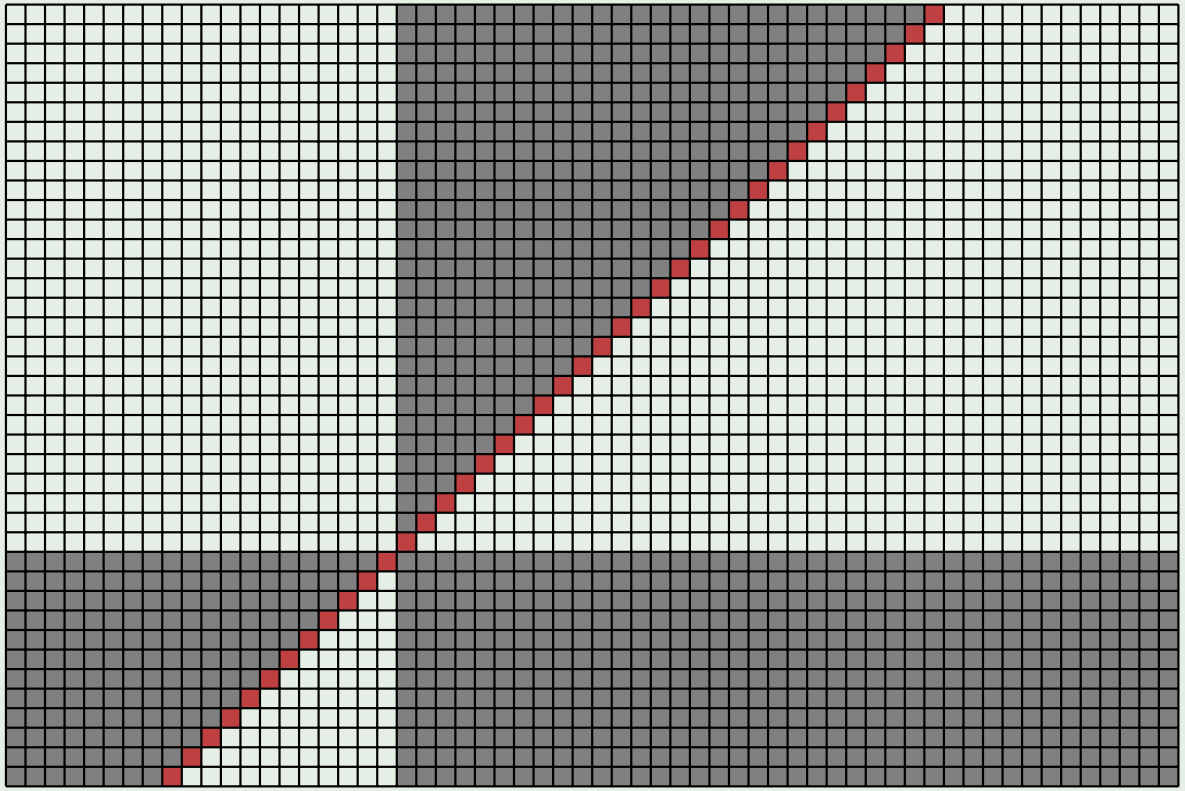

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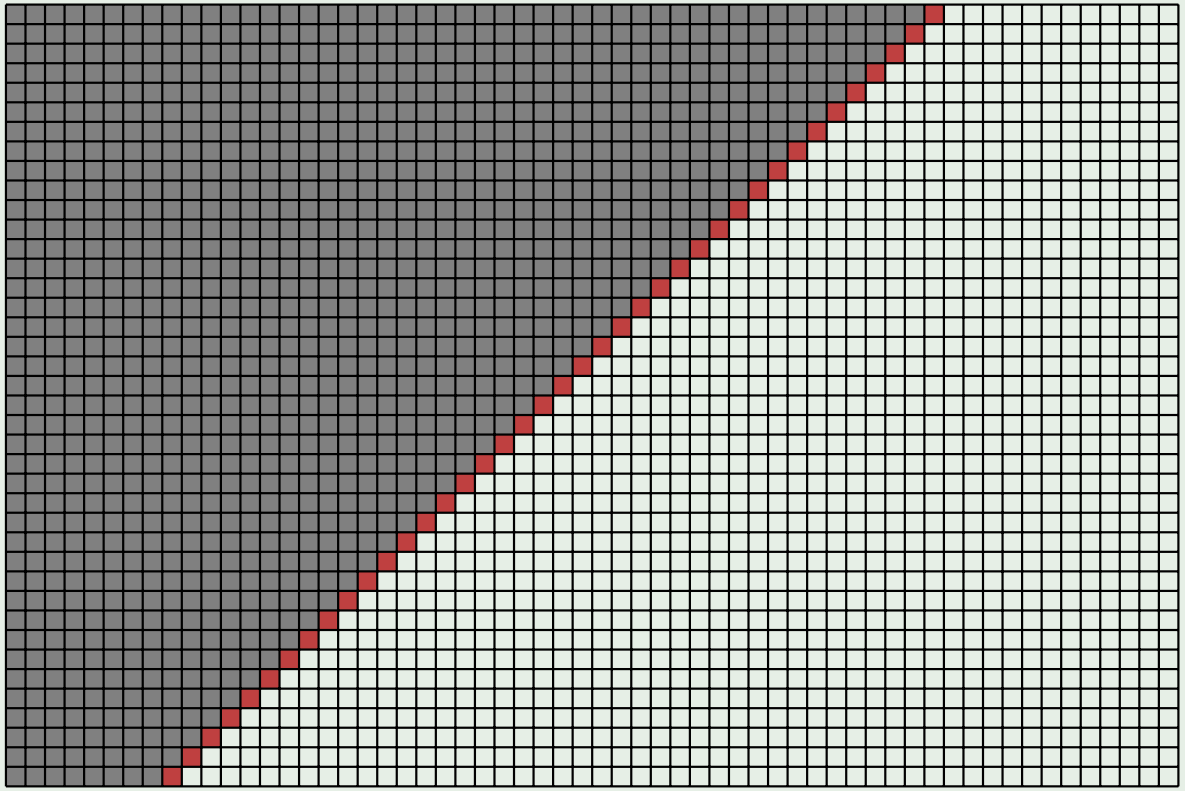

Countable 2D SFT with infinitely many periodic orbits.

Some topology

- Topology of $\Sigma^{\mathbb{Z}^d}$ is generated by *cylinder sets* $[P] = \{x \in \Sigma^{\mathbb{Z}^d} : x|_D = P\}$ for finite patterns $P \in \Sigma^D$
- A configuration $x \in X$ of shift space is isolated iff there is a finite pattern $P \in \Sigma^D$ st. $X \cap [P] = \{x\}$
- We say P isolates x in X

Example: Computability of isolated points

Proposition

If $X \subset \Sigma^{\mathbb{Z}^d}$ is an SFT, then each of its isolated configurations is computable

Proof.

Let $x \in X$ be isolated by a finite pattern P. For $n \in \mathbb{N}$, consider all possible ways of extending P to $[-n, n]^d$ that avoid the forbidden patterns of X. Since P isolates x in X, the symbol at each coordinate $\vec{v} \in \mathbb{Z}^d$ is eventually fixed to $x_{\vec{v}}$. This gives an algorithm to compute $x_{\vec{v}}$ from $\vec{v}$. $\square$

Example: Aperiodic implies uncountable

Proposition

A countable shift space X contains a fully periodic configuration

Proof.

Let $Y = \overline{\mathcal{O}(y)} \subseteq X$ be a minimal subsystem. We claim that Y is finite. If Y is infinite, then the set of limit points of $\mathcal{O}(y)$ is all of Y, so Y is perfect. A nonempty perfect complete metric space is uncountable, a contradiction. $\square$

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Theorem (Ballier, Durand, Jeandel 2008)

An infinite countable 2D SFT contains a configuration with exactly one direction of periodicity

More topology

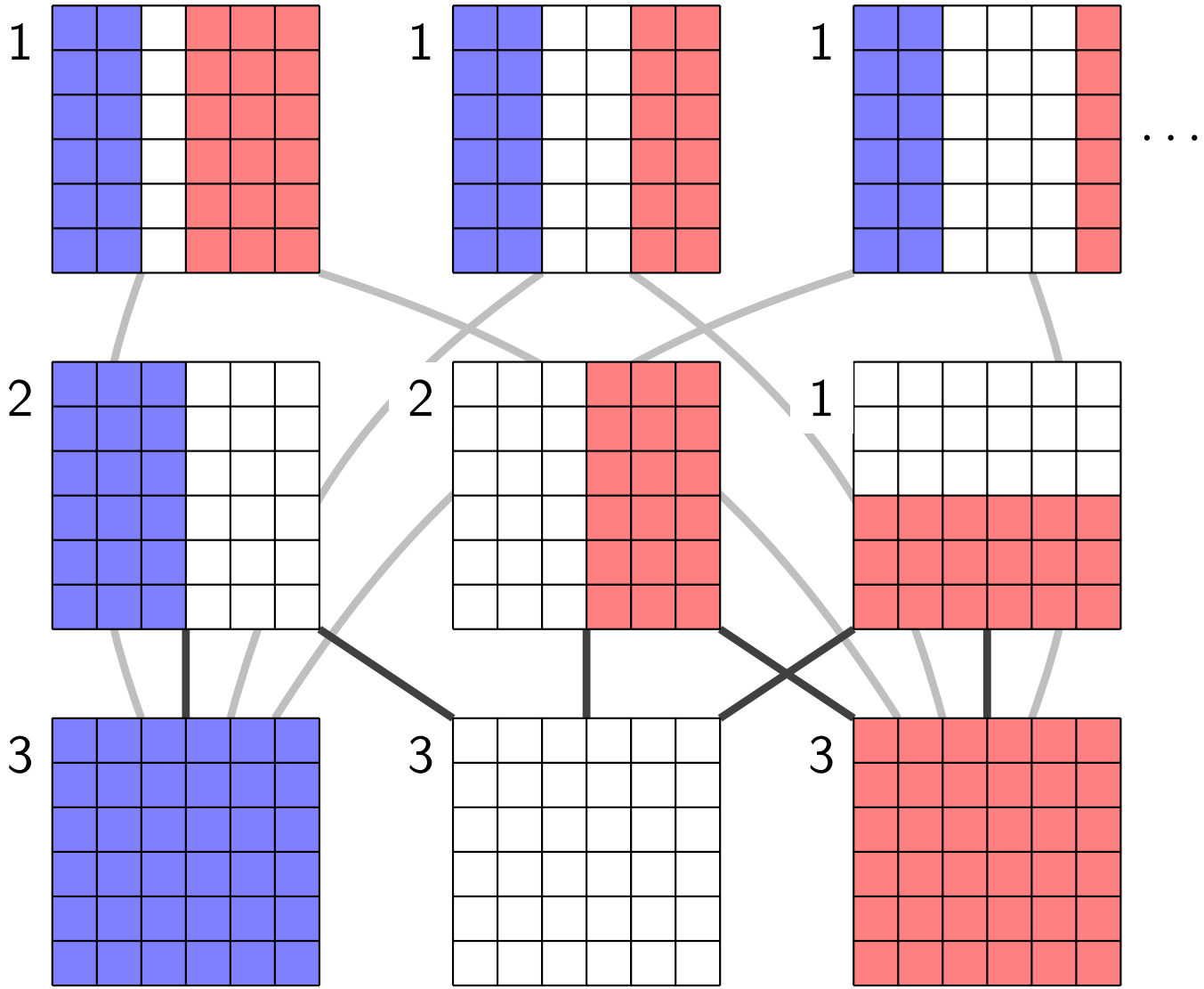
- *Cantor-Bendixson derivative* of X is $X' = \{x \in X : x \text{ not isolated in } X\}$
- Transfinite iteration: $X^{(0)} = X$, $X^{(\alpha+1)} = (X^{(\alpha)})'$, $X^{(\lambda)} = \bigcap_{\beta<\lambda} X^{(\beta)}$ for limit ordinal λ
- CB-rank $r(X)$ is lowest ordinal λ with $X^{(\lambda+1)} = X^{(\lambda)}$
- A shift space X is countable iff $X^{(r(X))} = \emptyset$
- Rank $\rho(x)$ of $x \in X$ is lowest ordinal λ with $x \notin X^{(\lambda)}$

Even more topology

- Subpattern order: $x \prec y$ if $\overline{\mathcal{O}(x)} \subsetneq \overline{\mathcal{O}(y)}$ (y contains all finite patterns of x)
- Subpattern poset $P(X)$: orbit closures $\overline{\mathcal{O}(x)}$ for $x \in X$, ordered by inclusion, as abstract poset
- If $x \prec y$, then $\rho(x) > \rho(y)$ (if the ranks exist)
- $\prec$-minimal element $\Leftrightarrow$ minimal orbit closure
- $\prec$-maximal elements exist in all shift spaces

Subpattern order

Example of ranks and order $\prec$ in a countable SFT



Structure of countable SFTs

Theorem (Ballier, Jeandel 2013)

In a rank-4 countable 2D SFT, configurations of rank

- 4 *are doubly periodic*
- 3 *are periodic*
- 2 *are periodic or 'star-shaped'*
- 1 *are computable*

Theorem (T. 2015)

There is a rank-5 countable SFTs with an uncomputable configuration

Proved by embedding computation into the SFT

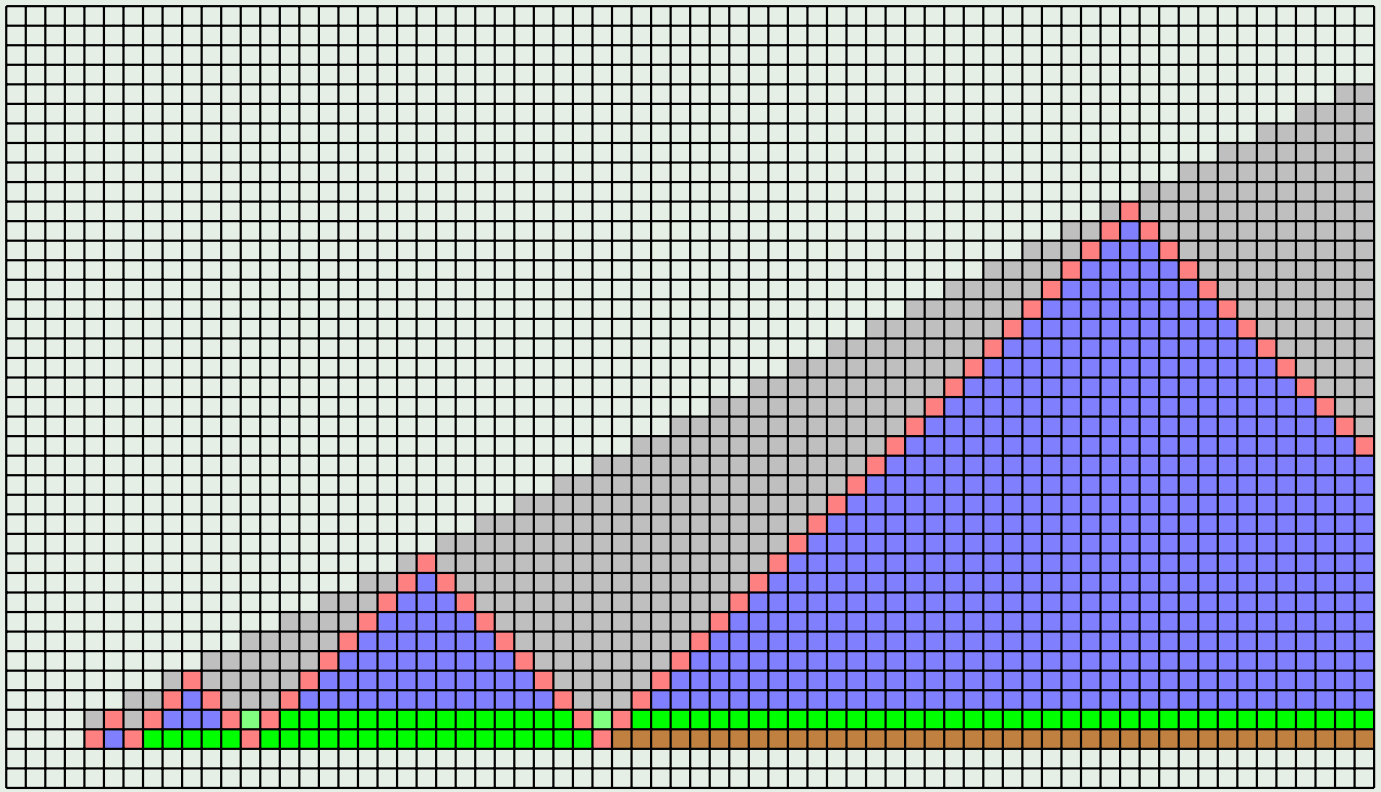
Computation

- How to embed computation into countable SFT?
- Infinite tape of Turing machine $\Rightarrow$ uncountability
- *Counter machine*: finite state set, fixed finite number of counters with values in $\mathbb{N}$
- Can increment/decrement any counter and check if it holds the value 0
- Counter machines are computationally universal

Computation

Example (Simulation of computation)

Simulating a counter machine in a countable SFT:



Ascending chains

Theorem (Ballier, Durand, Jeandel 2008)

If X is a countable shift space, then $P(X)$ contains no infinite ascending chains

Proof.

The subshift $Y = X^{(r(X))} \subseteq X$ is a perfect space. Since X is countable, $Y = \emptyset$, and thus every configuration $x \in X$ has a rank $\rho(x)$. Recall that $x \prec y$ implies $\rho(x) > \rho(y)$. An infinite ascending chain in $P(X)$ would give an infinite descending chain of ordinals, which is impossible. $\square$

Descending chains

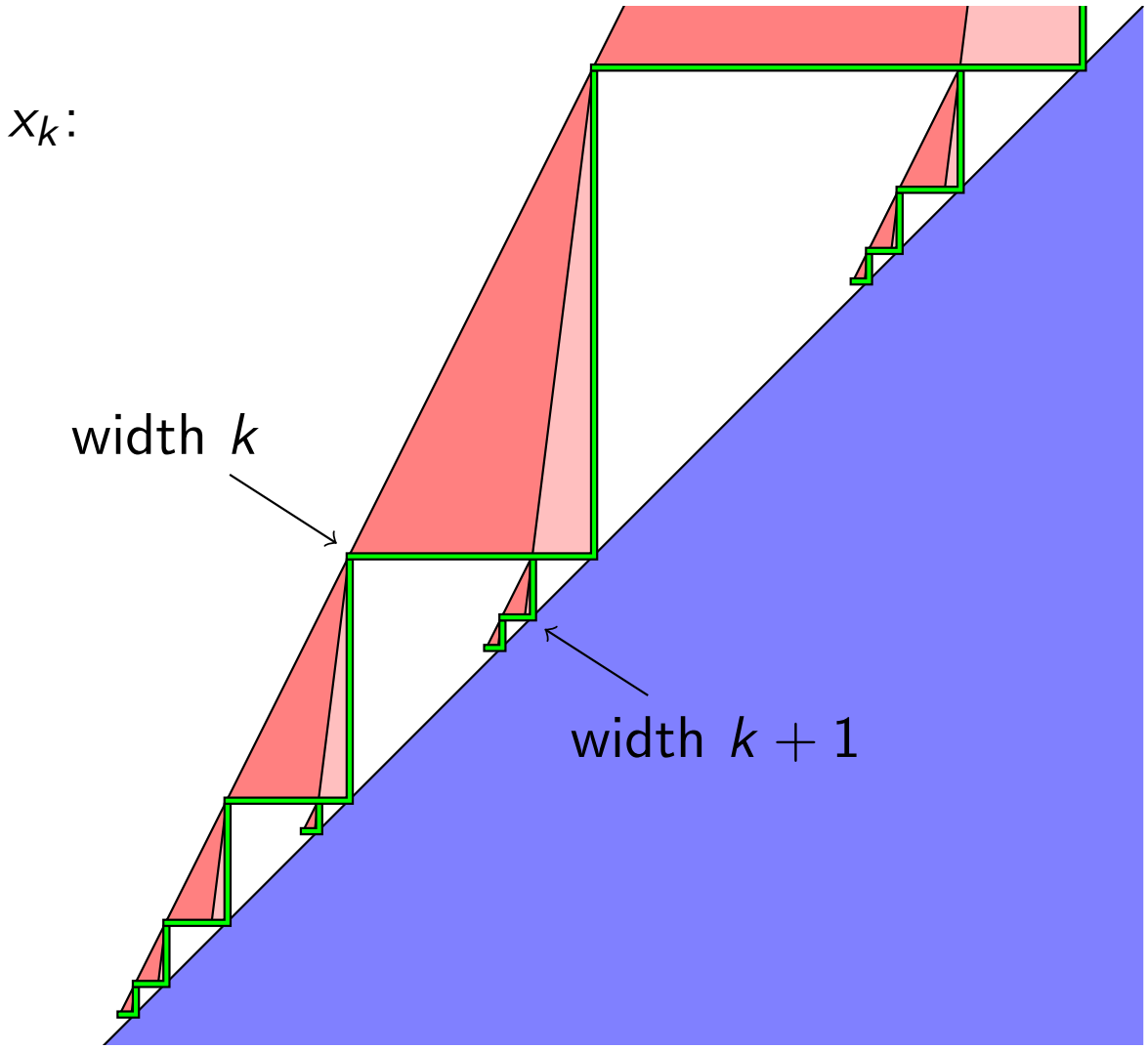
Theorem (Salo, T. 2013)

There exists a countable 2D SFT X such that $P(X)$ contains an infinite descending chain

Construction involves a hierarchical structure different from the one present in Robinson's tiles, substitutions etc.

Descending chains

Embedding a chain $x_1 \succ x_2 \succ x_3 \succ x_4 \succ \cdots$



Hyperarithmetical posets

- A set $Q \subset \mathbb{N}$ is *hyperarithmetical*, if it is the unique infinite branch of a computable directed tree on $\mathbb{N}$
- A *hyperarithmetical poset* is a poset $(Q, \leq)$ such that $Q \subset \mathbb{N}$ and $\leq \subset Q \times Q$ are hyperarithmetical
- $P(X)$ is hyperarithmetical for every countable SFT X

Theorem (T. 2015)

For each hyperarithmetical poset $(Q, \leq)$ with no infinite ascending chain, there exists countable 2D SFT X and order-embedding $\phi : (Q, \leq) \to P(X)$ such that all elements of $P(X) \setminus \phi(Q)$ have height ≤ 3

Growth rates of shift spaces

- Pattern complexity of shift space X is the number of distinct size-n blocks: $P_n(X) = \#\{x|_{[1,n]^d} : x \in X\}$
- Topological entropy: $h(X) = \lim_n \frac{\log P_n(X)}{n^d}$
- Countable shift spaces have zero entropy (equivalently, $P_n(X)$ grows sub-exponentially in n^d)
- Pattern complexity of an SFT is upper semi-computable

Growth rates of countable SFTs

Theorem (T. 2017, unpublished)

Given a sub-exponential upper semi-computable function $f : \mathbb{N} \to \mathbb{N}$, there exists a countable 2D SFT X with $P_n(X) > f(n^2)$ for all $n \geq 1$

Something stronger is probably also true.

Open problems

- High-level structure of subpattern posets has been characterized; what about the rest?
- What are the possible CB-ranks of countable SFTs?
  - [Ballier, Jeandel 2013] It cannot be $\lambda + n$ for a limit ordinal λ and $n \in \{0, 1, 2\}$
  - [T. 2015] It can be $\lambda + 4$ for any computable ordinal λ
  - Is $\lambda + 3$ possible?
- Do all countable sofic shifts have countable SFT covers?