Embedding computations in tilings

Andrei Romashchenko

26 September 2017, ENS de Lyon

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  Turing machine with one or many bi-infinite tapes

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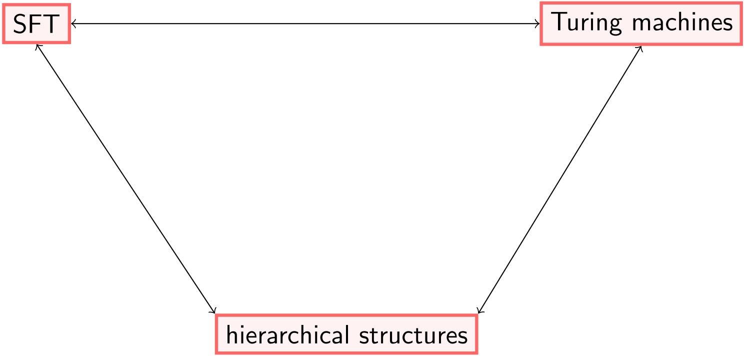
$$\boxed{\text{SFT}} \longleftrightarrow \boxed{\text{Turing machines}}$$


SFT
Turing machines
hierarchical structures

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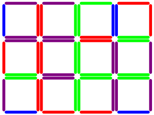

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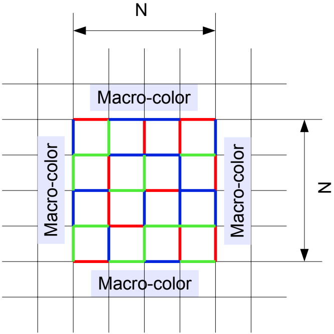
- observe that *every* **self-similar** tile set is aperiodic
- construct *some* **self-similar** tile set

Macro-tile:



an $N \times N$ square made of matching τ-tiles

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Definition 2. A tile set ρ is **simulated** by τ: there exists a family of τ-macro-tiles R such that

- R is *isomorphic* to ρ, and
- every τ-tiling can be *uniquely* split by an $N \times N$ grid into macro-tiles from R.

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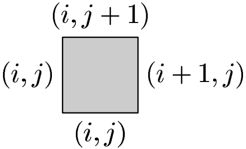

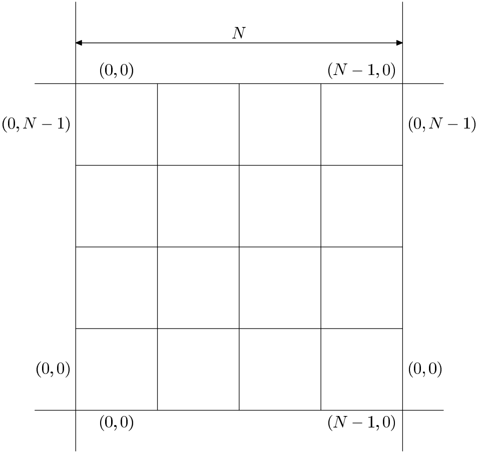
N
$(0,0)$
$(N-1,0)$
$(0,N-1)$
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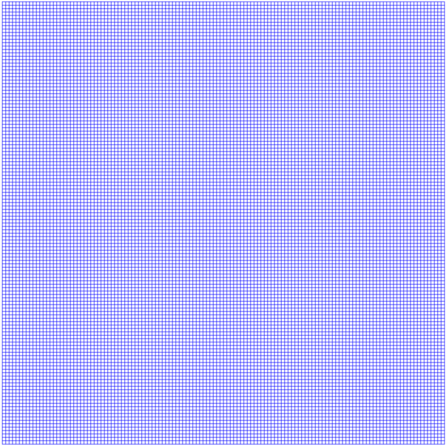

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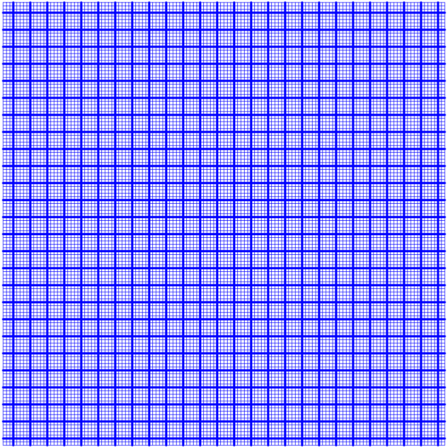

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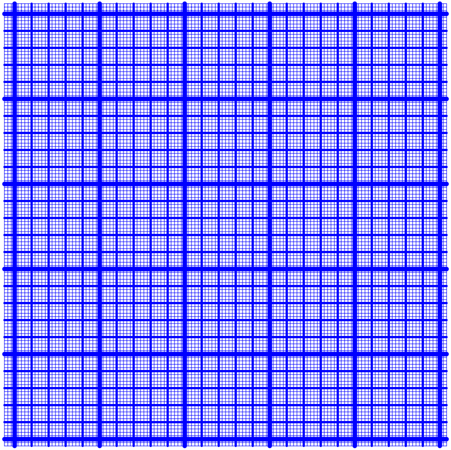

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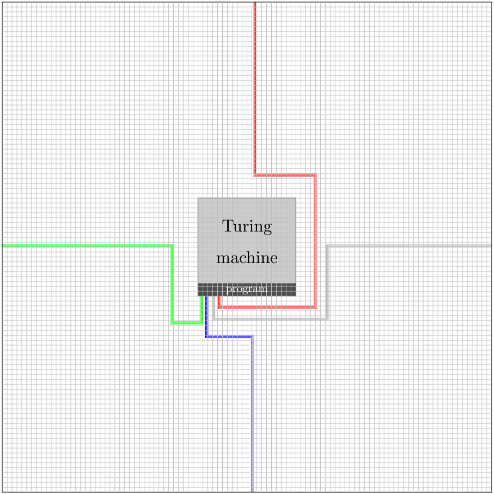
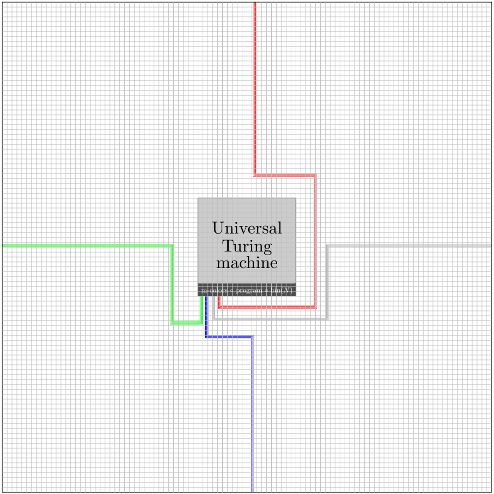
$\Downarrow$

a TM that accepts only 4-tuples of colors for the ρ-tiles

The scheme of implementation:



A more generic construction: universal TM + program



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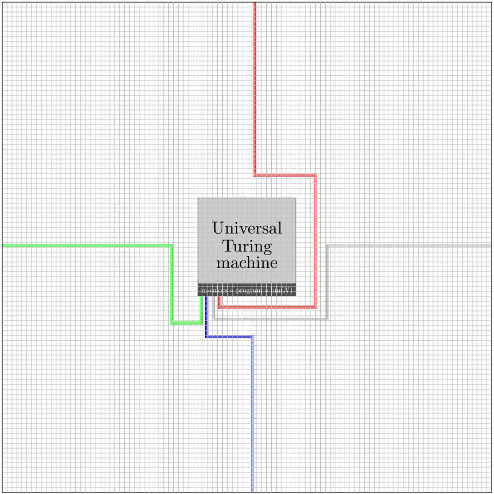
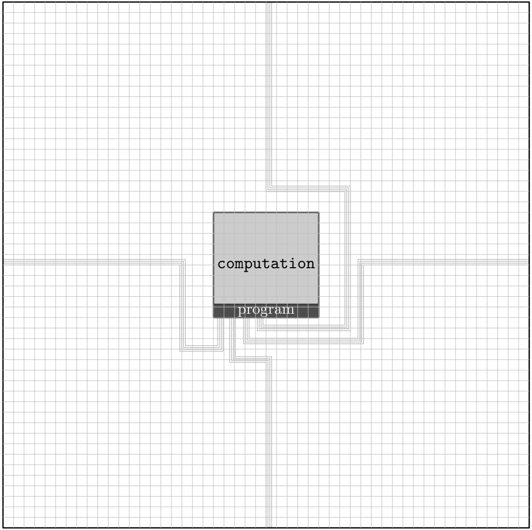


A fixed point: simulating tile set = simulated tile set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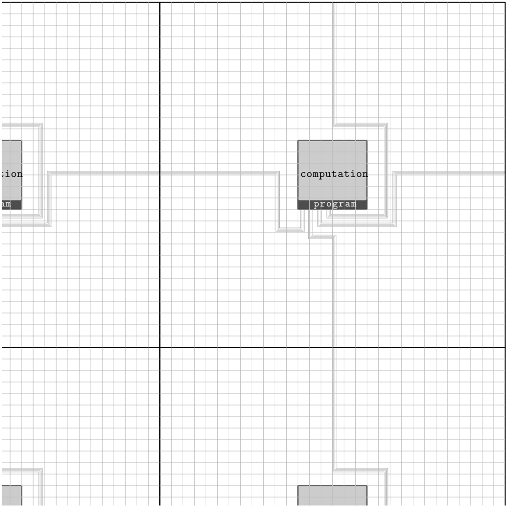

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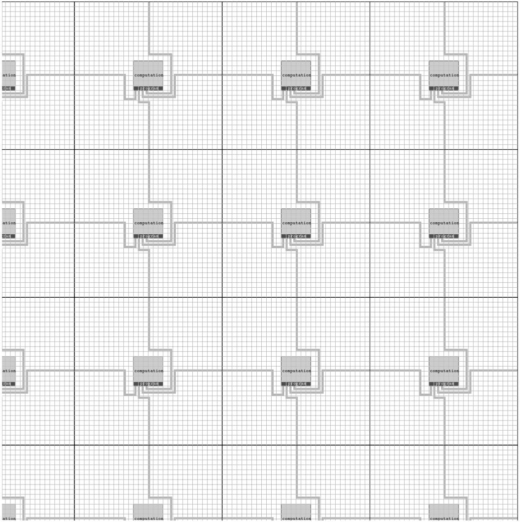
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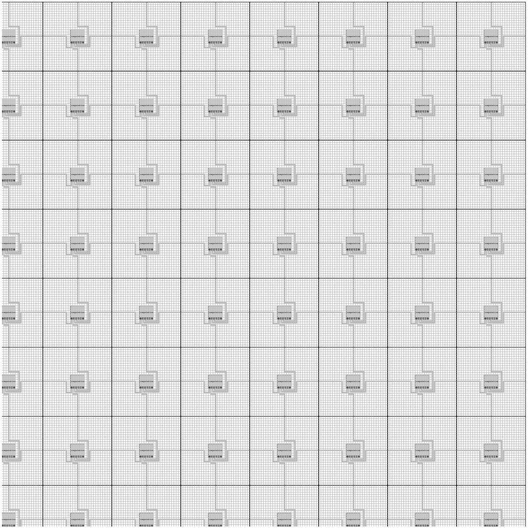

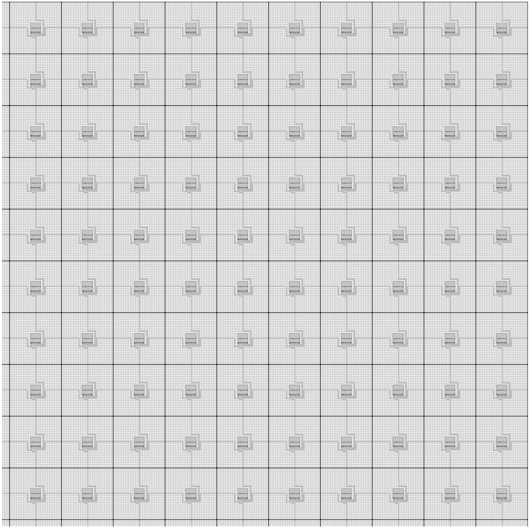

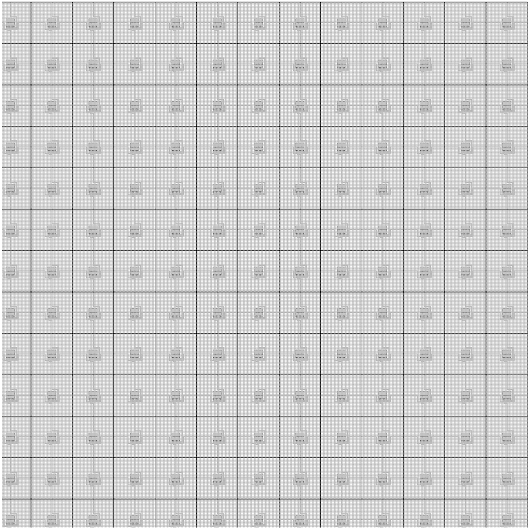

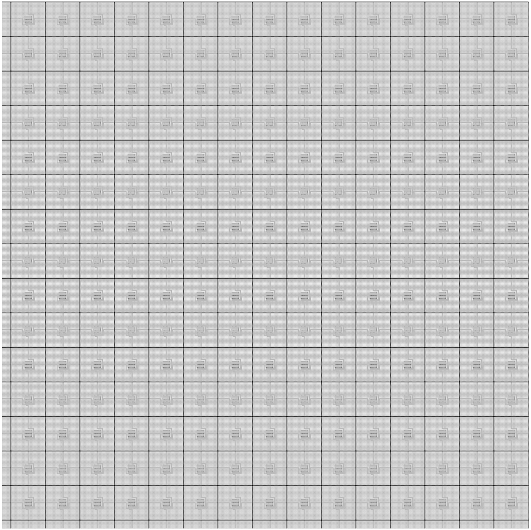

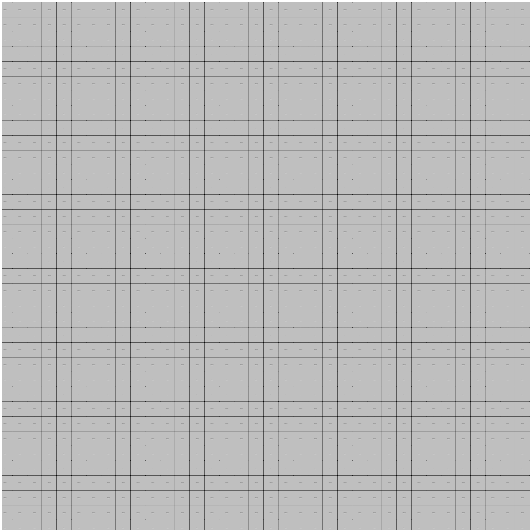

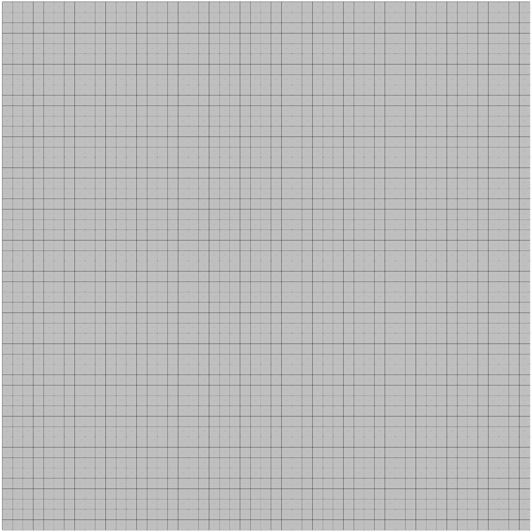
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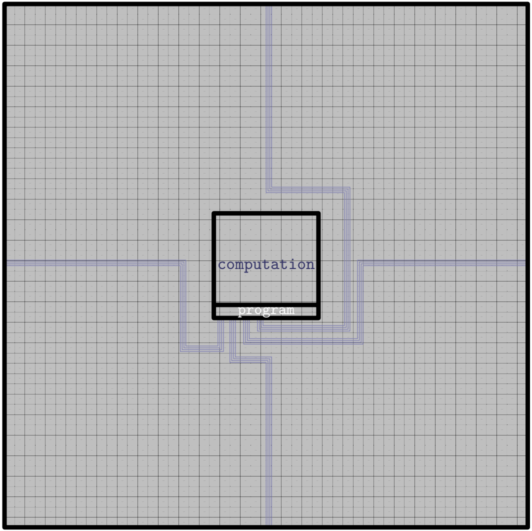
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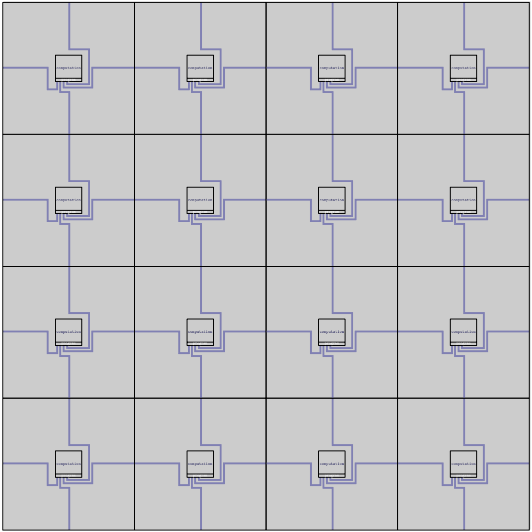
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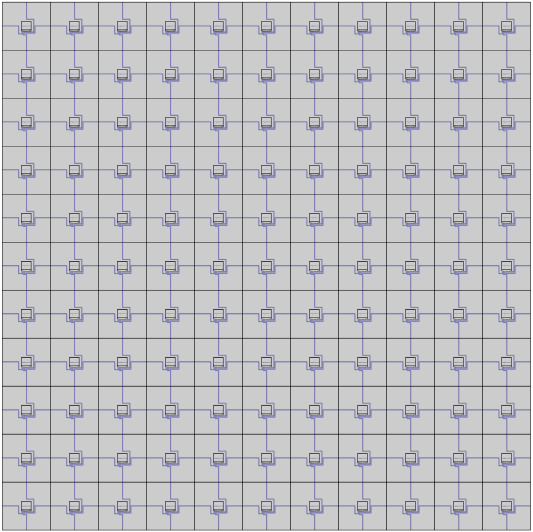

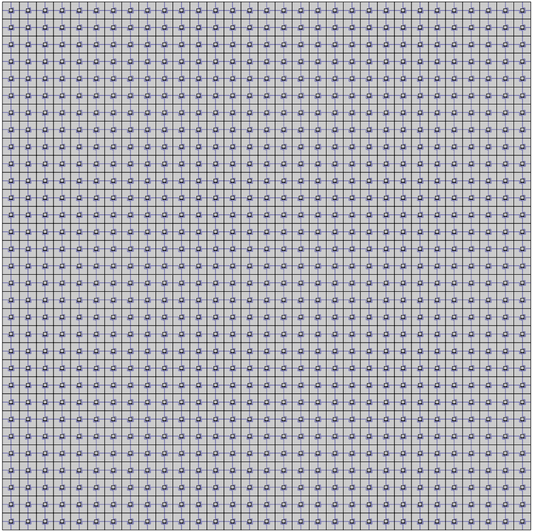

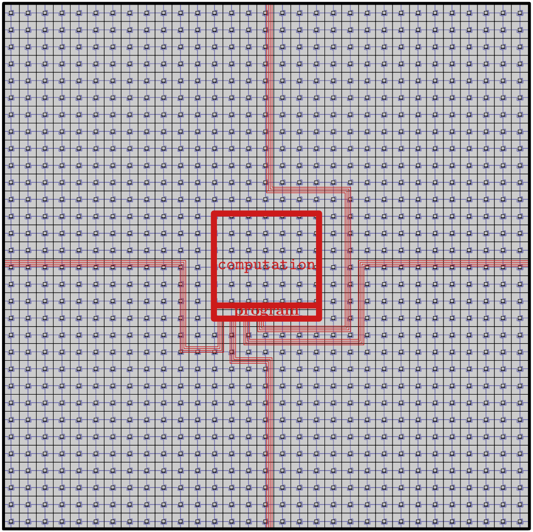
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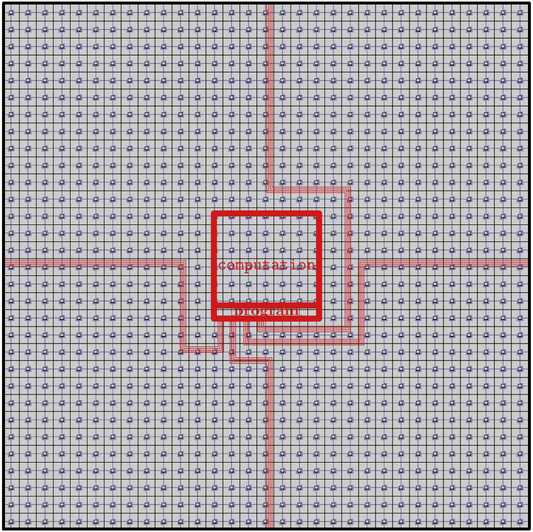

N.B. We can variate the zoom factor!

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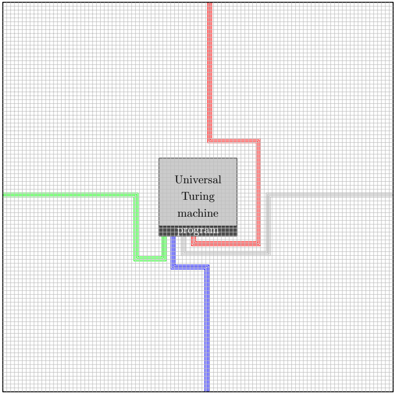
Ollinger: take Robinson's construction and remove everything that may appear *not* infinitely often

our plan: take the fixed-point construction and enforce everything that may appear at least *once*

How to get **aperiodicity** + **minimality** ?



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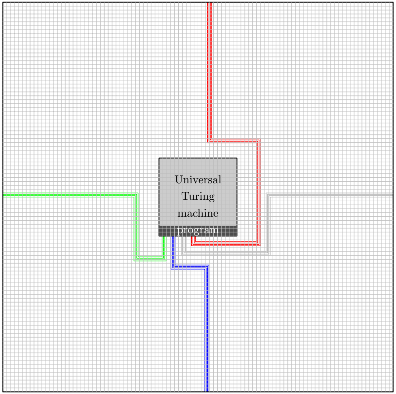
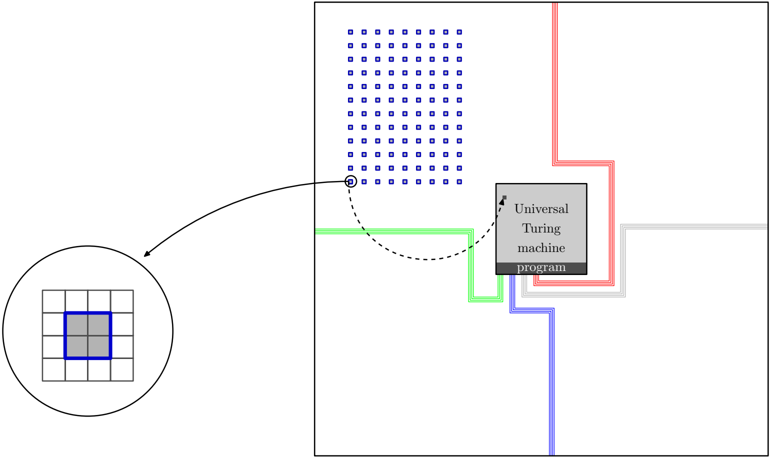


The problematic part is the computation zone...

Duplicate each 2×2 pattern that *may* appear in the computation zone!



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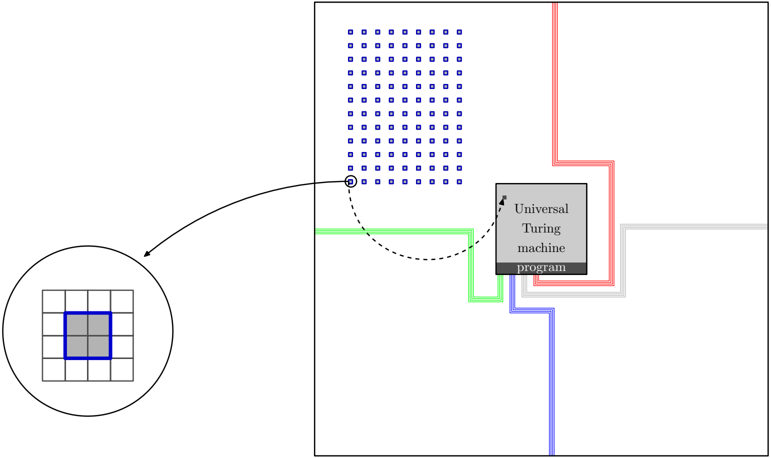
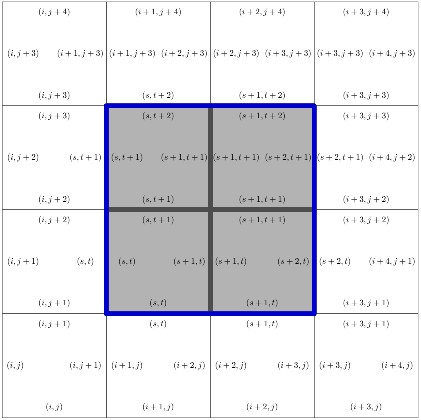


Imprisonment for diversity!

A slot for a 2×2 patterns from the computation zone:



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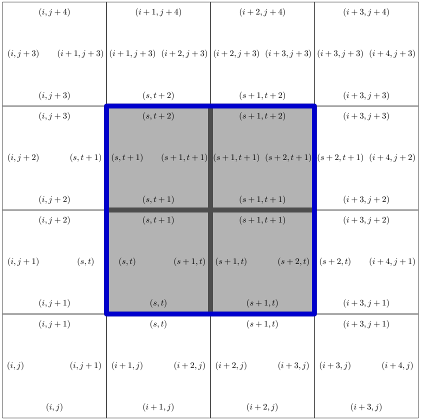

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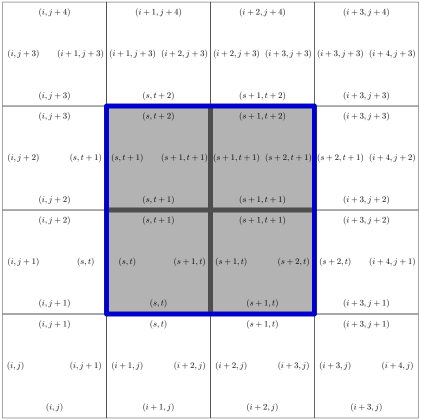

Grey cells: It seems we sit inside of the computation zone,
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White cells: Guys, you four are living in a small prison
in the middle of nowhere...

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Any number that can appear in real maths. And even slightly more.

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- **Schraudner:** What about some kind of (uniform) mixing?

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Transitive: There exists a configuration that contains all globally admissible finite patterns.

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- transitivity: random instantiation of red tiles

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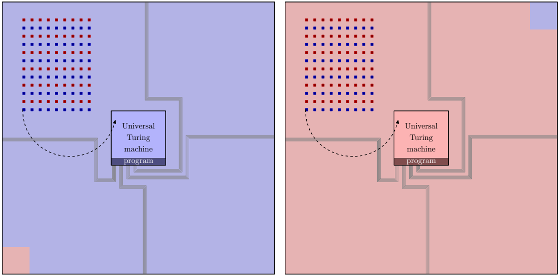

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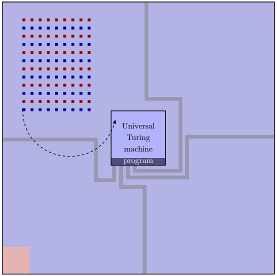

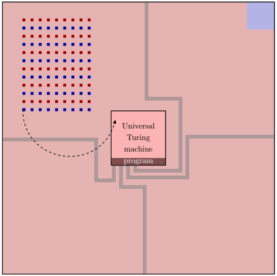

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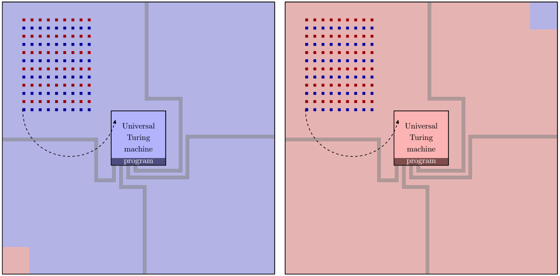

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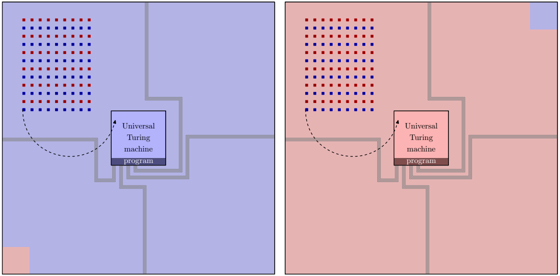

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- the fraction red tiles in a red macro-tile approaches the limit

Hierarchical constructions: self-simulating vs Berger/Robinson's

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	self-simulating tilings	Robinson's construction
undecidability of the domino problem	+	Berger'66
tilings with only noncomputable points	+	Hanf–Myers'74
any *effectively closed* 1D subshift is isomorphic to the subdynamics of a 2D SFT	Durand–R–Shen	Aubrun–Sablik
similar result for *minimal* subshifts	Durand–R	?
strongly deterministic tilings	–	Kari, Papasoglu, Lukkarila, Le Gloannec, Ollinger
pairwise different black squares in the white ocean	Westrick	?

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- embedded simple finite automata (transducers),
- aperiodic tilings with *very small* number of tiles
- tilings with really interesting properties

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Thank you!