Substitutions and Strongly Deterministic Tilesets

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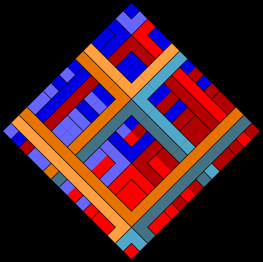
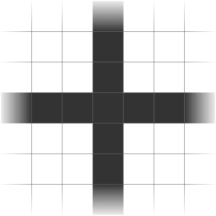
Aperiodic 2017, Lyon
September 26, 2017



Recognizing families of colorings

$\Sigma = \{\square, \blacksquare\}$



Family of Σ-colorings
(*subshift*)

We recognize colorings of the discrete plane via local constraints.

Theorem [Mozes 89]. Colorings "generated by" (expansive) substitutions are "recognizable".

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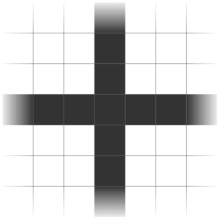

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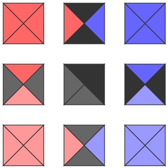

Set of local rules (*Wang tiles*)

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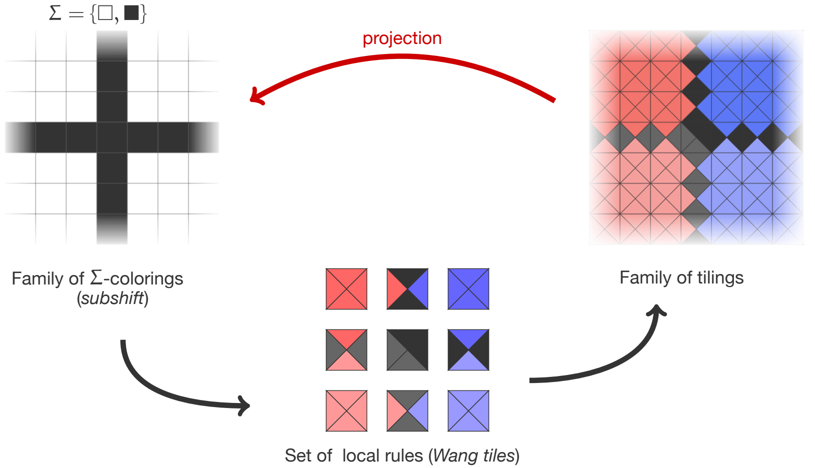

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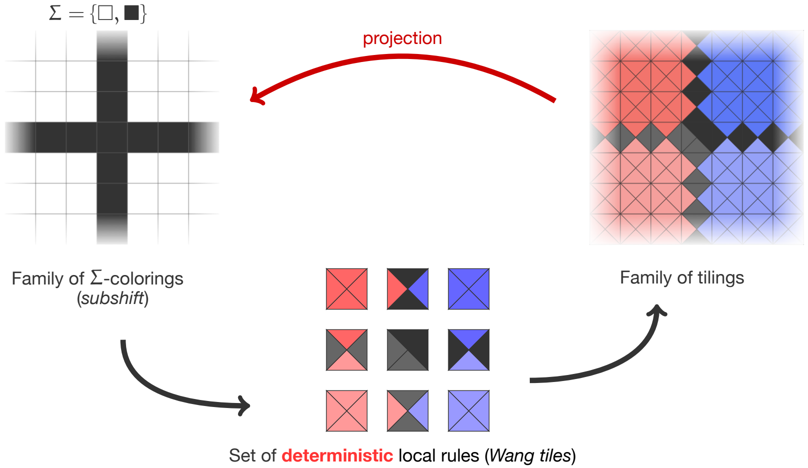


We recognize colorings of the discrete plane via local constraints.

Theorem[in CiE'12]. Colorings "generated by" 2×2 substitutions are "**deterministically** recognizable".

1. Tilings

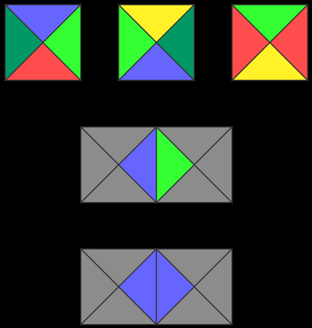
Tilings by Wang tiles

A **Wang tile** is an oriented (no rotations allowed) unit square tile carrying a **color on each side**.

A **tileset** τ is a finite set of Wang tiles.

A **configuration** $c \in \tau^{\mathbb{Z}^2}$ associates a tile to each cell of the discrete plane $\mathbb{Z}^2$.

A **tiling** is a configuration where the colors of the common sides of neighboring tiles match.



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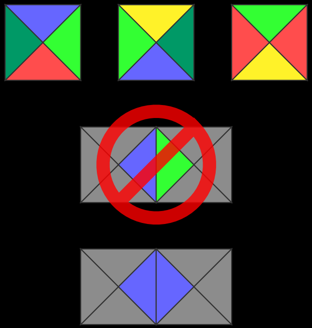
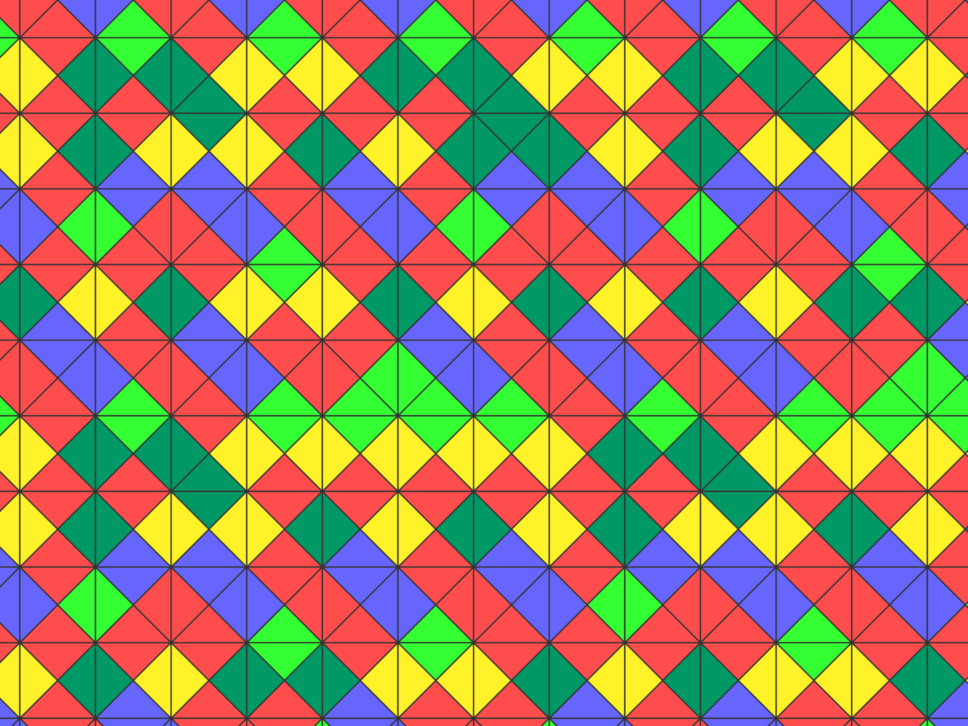




Historical context

In the early 60s, H. Wang reduces the decidability problem of the $\forall\exists\forall$ class of **first order logic formulas** to a question of discrete tiling.

***Domino Problem*, DP [Wang, 1961].** Given a tileset, is it possible to tile the plane?

Theorem [Berger, 1964]. The *Domino Problem* is undecidable.

Proof by construction of an aperiodic tileset describing a self-similar structure able to contain some **Turing computations**.

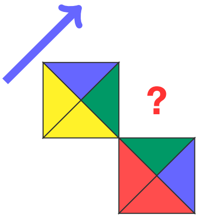
Aperiodicity. A tileset is **aperiodic** if it tiles the plane, but never in a periodic way.

Deterministic tilesets

Introduced by J. Kari in 1991 to prove the undecidability of the nilpotency problem for 1D cellular automata.

Notations: *NW* for North-West, *SE* pour South-East…

Deterministic tileset. A tileset τ is **NE-deterministic** if for any pair of tiles $(t_W, t_S) \in \tau^2$, there exists **at most one** tile t compatible to the west with t_W and to the south with t_S.



Partial local map.

$$f : \tau^2 \longrightarrow \tau$$

We symmetrically define **{NW,SE,SW}-determinism**.

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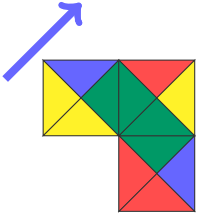

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$$f : \tau^2 \to \tau$$

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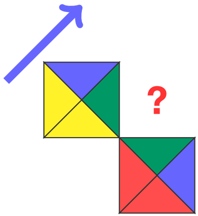
Deterministic tilesets

Bideterminism. A tileset is **bideterministic** if it is simultaneously deterministic in **two opposite directions**: NE & SW, or NW & SE.



Strong determinism. A tileset is **strongly deterministic** *(4-way deterministic)* if it is simultaneously deterministic in the **four directions** NE, NW, SW and SE.

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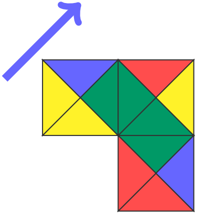

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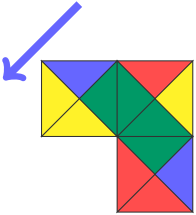

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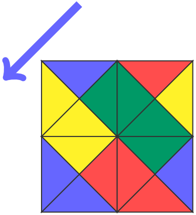

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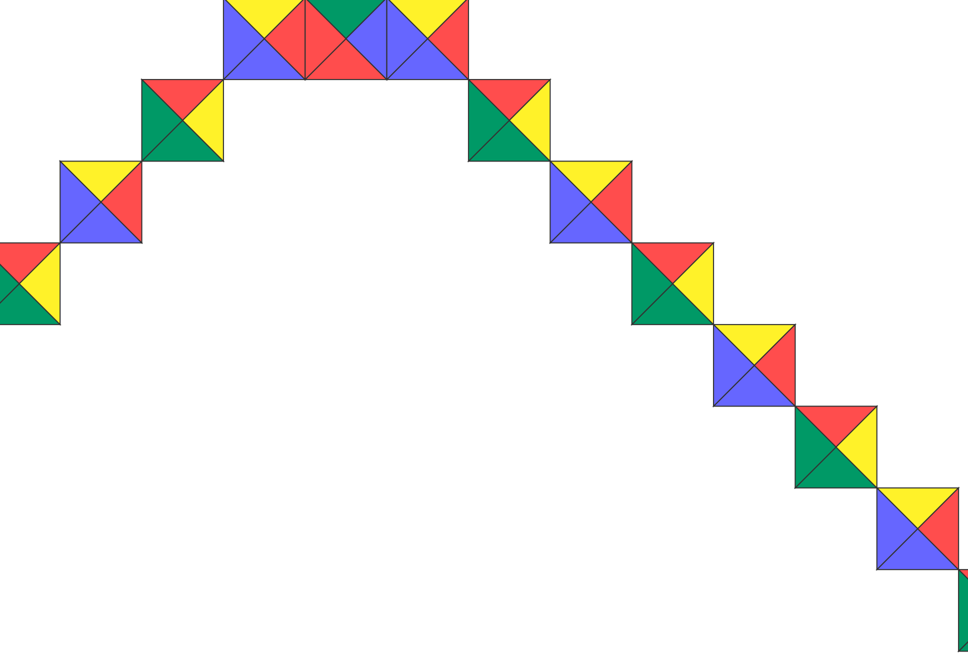
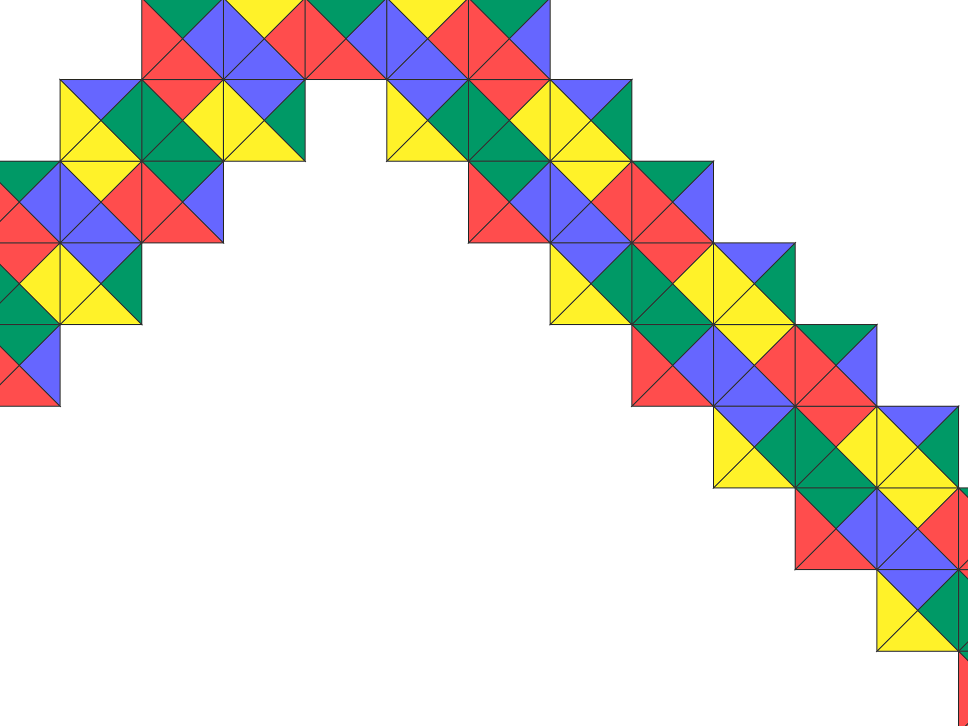
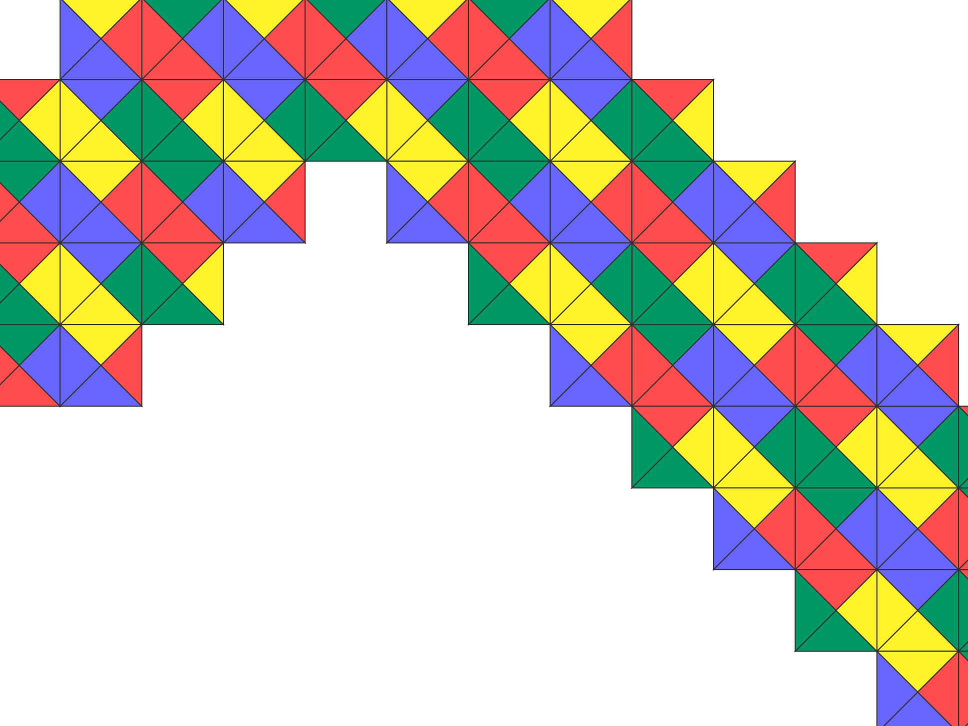
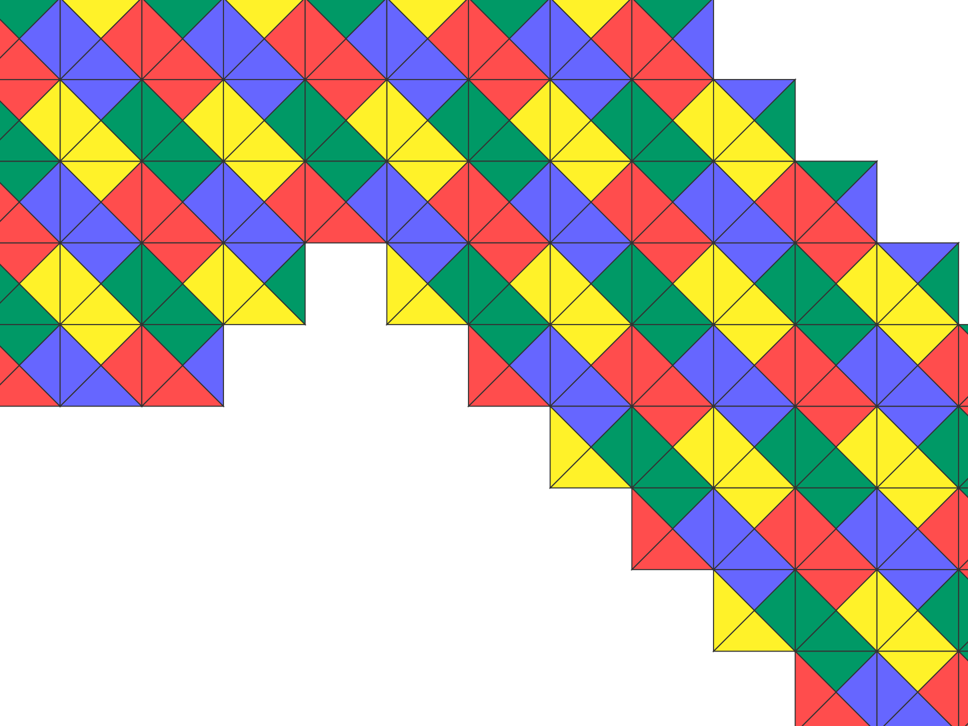
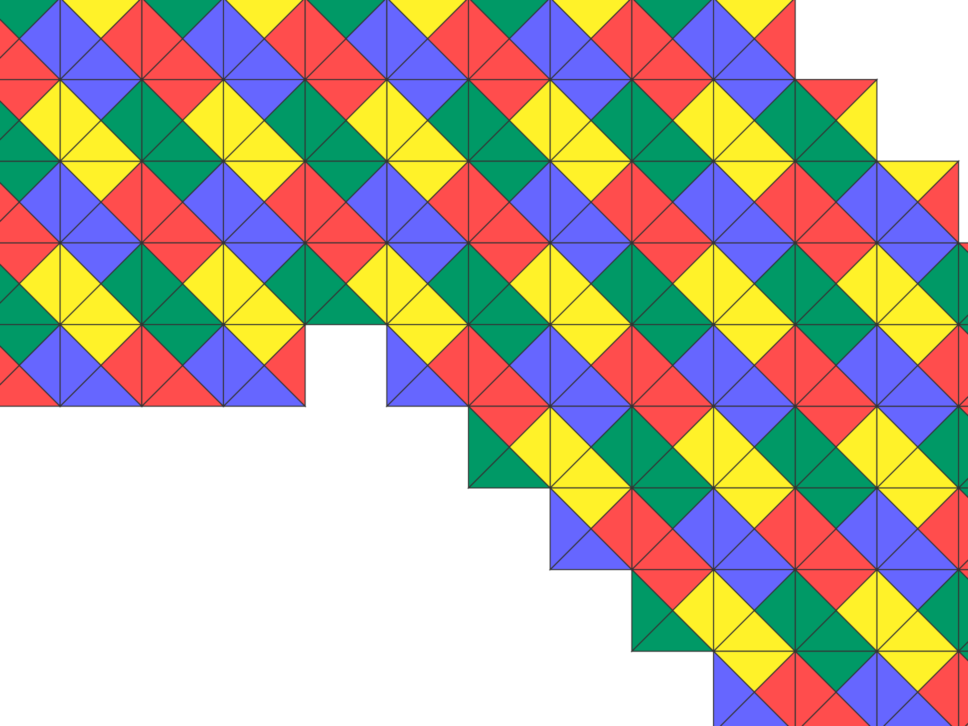
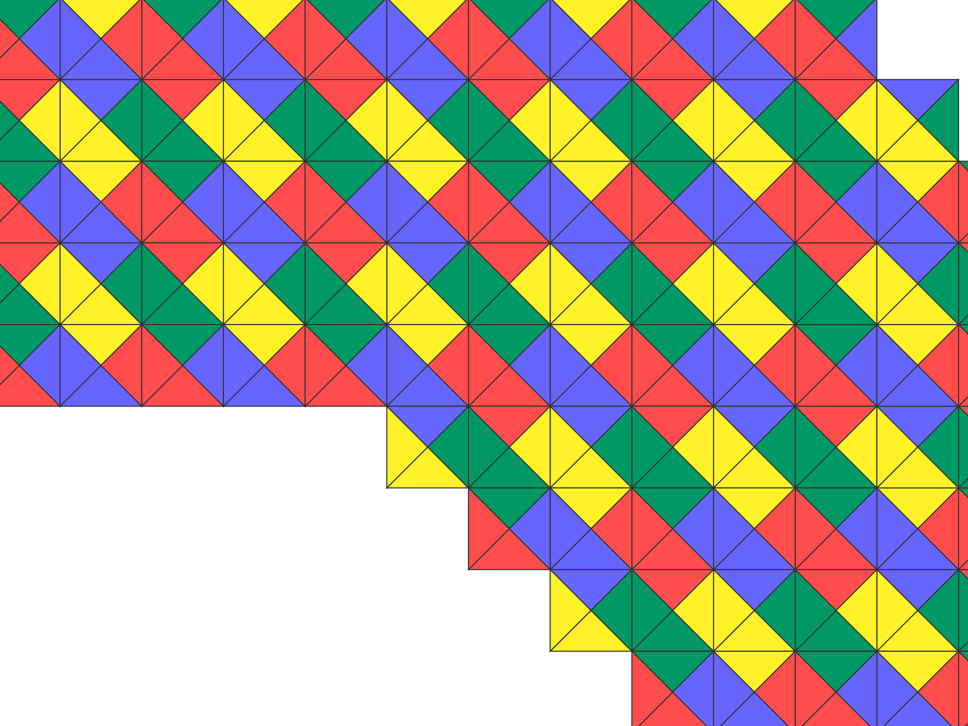


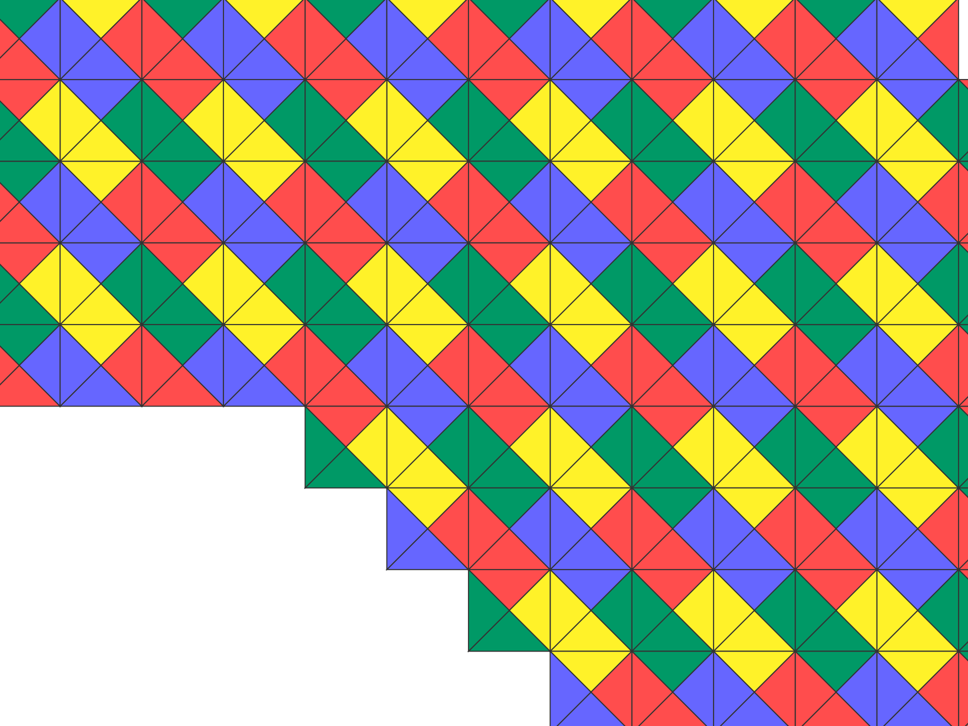


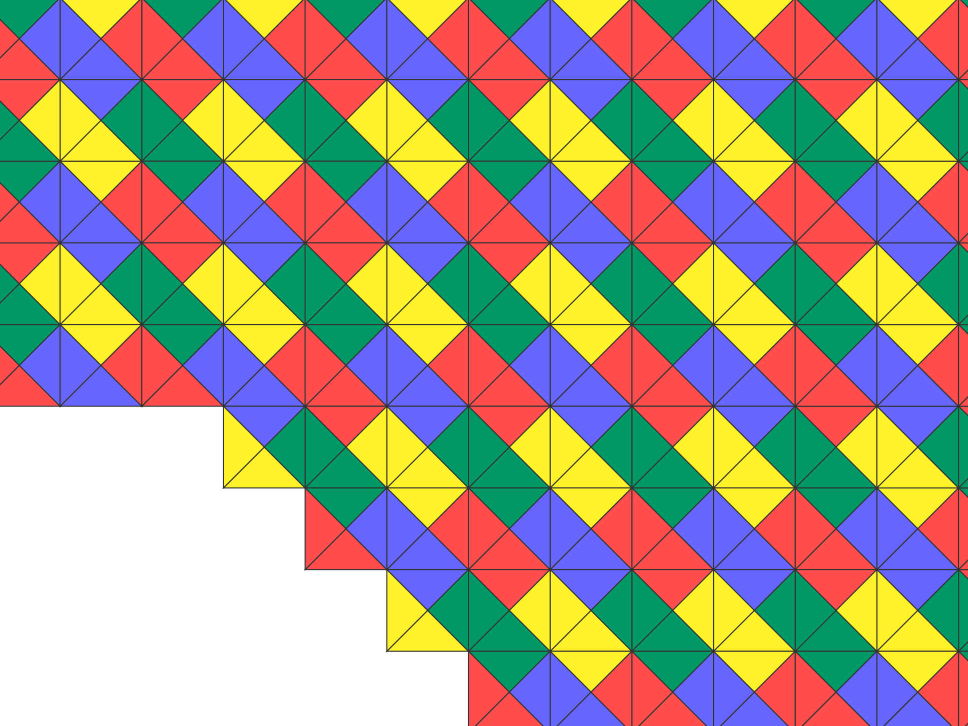


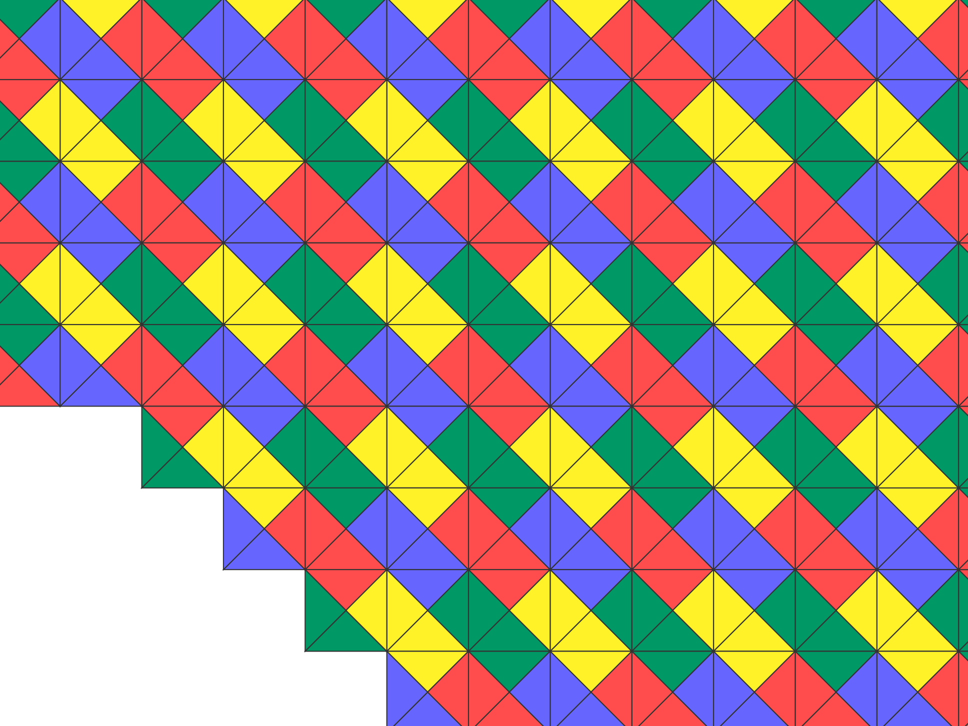


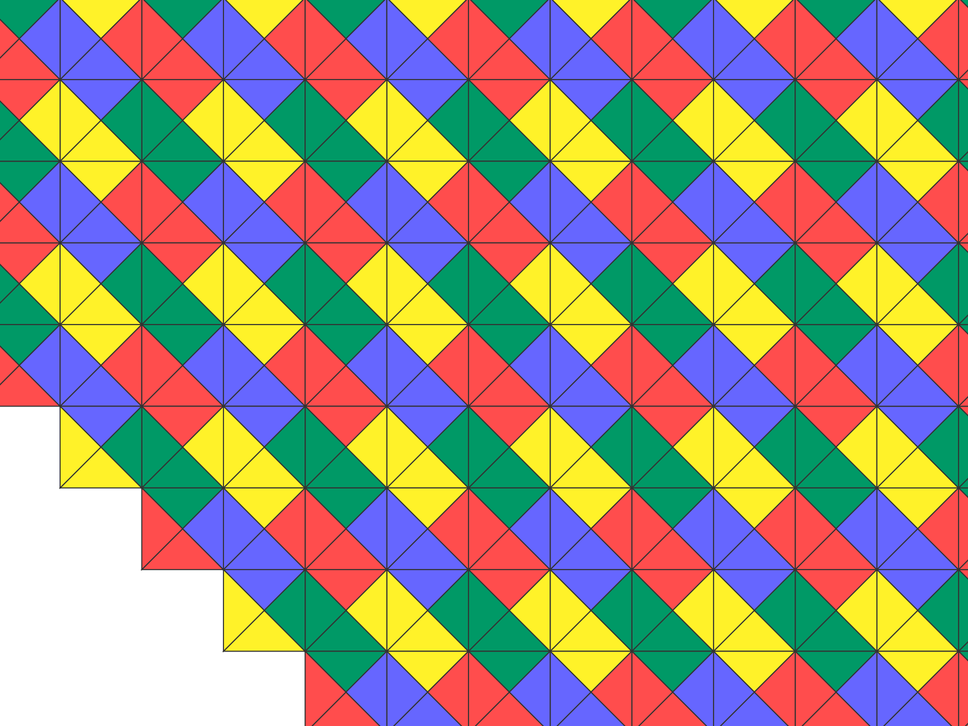


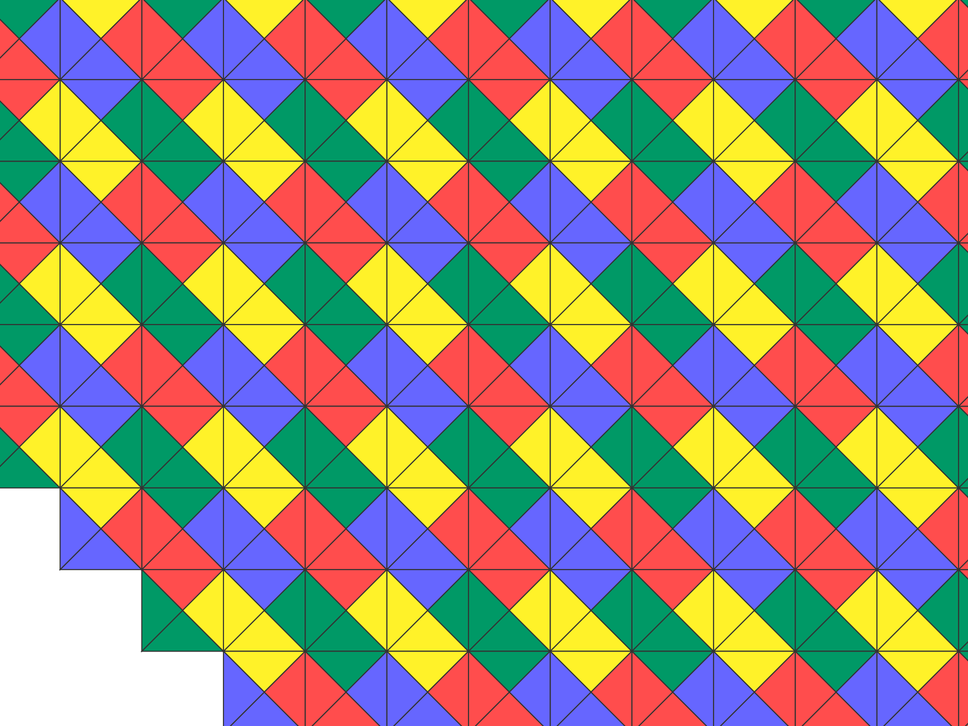


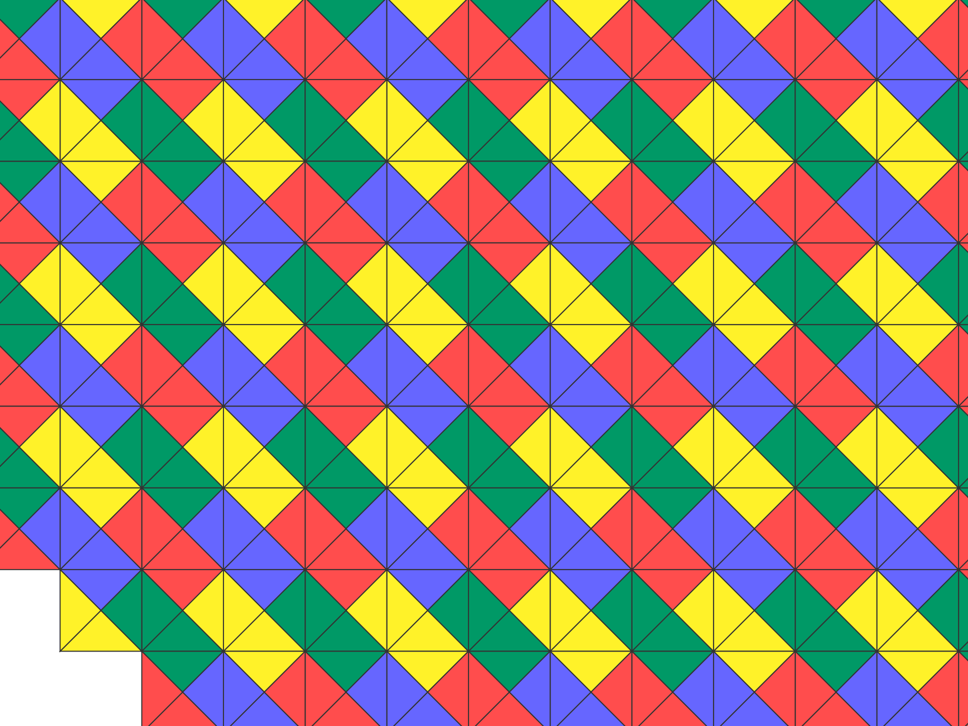


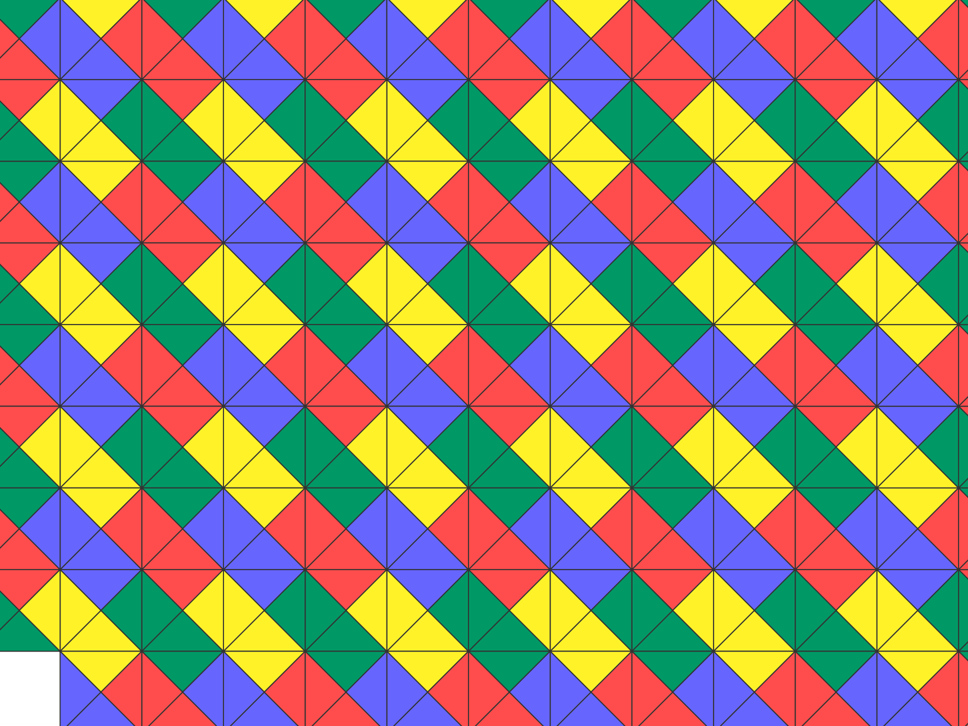


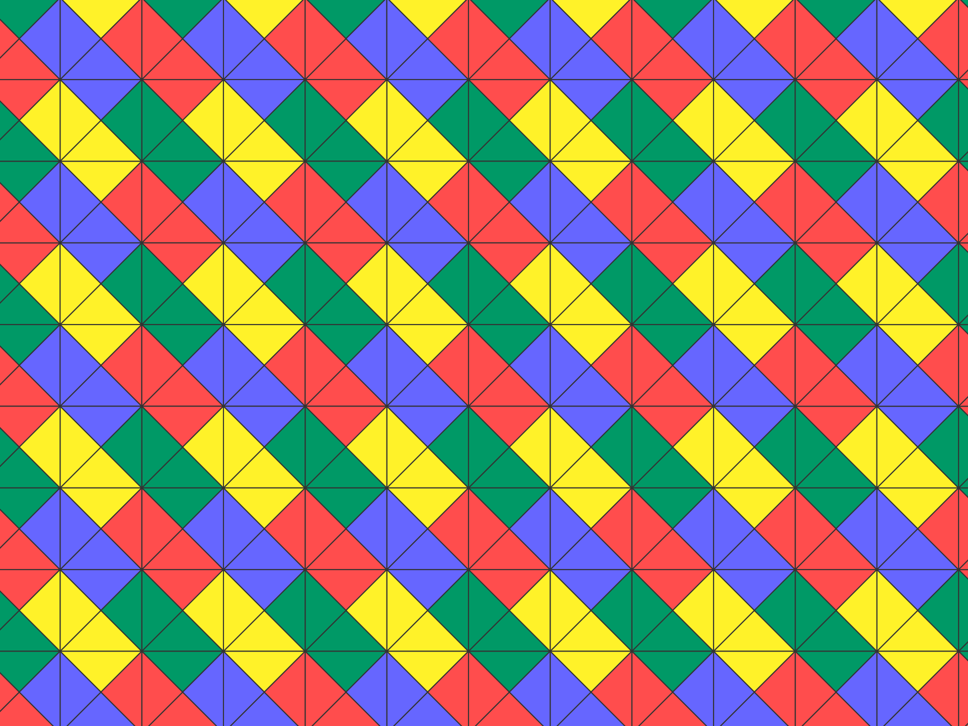


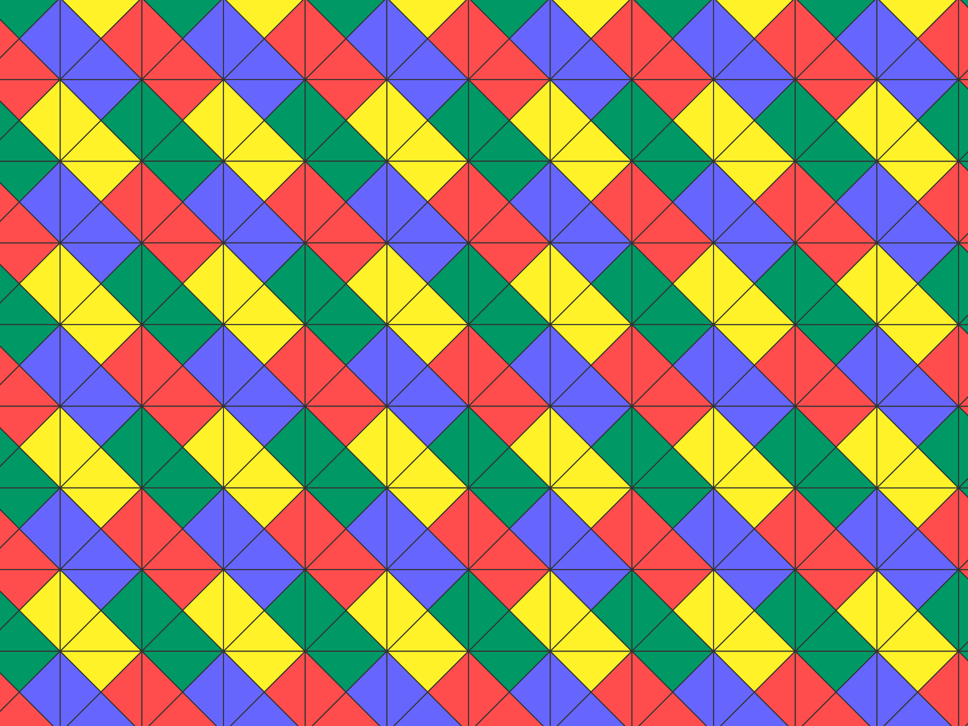


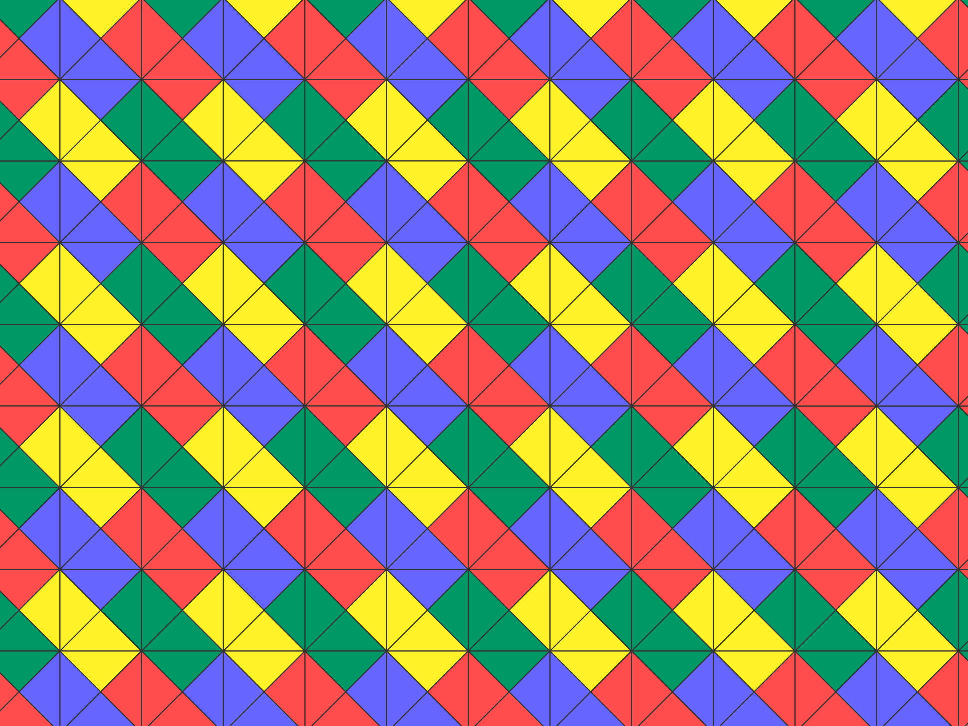












Deterministic tilesets: a short history

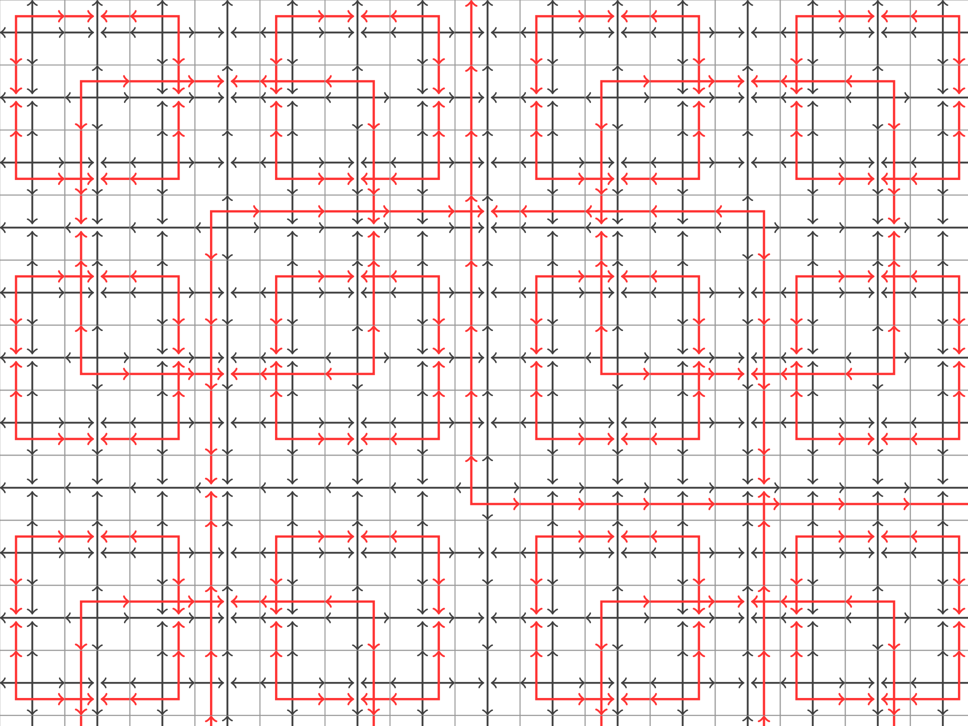
[Kari, 1991] introduced a (bi)determinization of **[Robinson, 1971]** to treat the nilpotency problem for cellular automata in dimension 1 (**Nil1D**)…Already proven in **[Aanderaa-Lewis 1974]**?!

Theorem [Kari, 1991; Aanderaa-Lewis, 1974]. **Nil1D** is undecidable.

Theorem [Kari, 1991; Aanderaa-Lewis, 1974]. There exist some **(bi)deterministic** aperiodic tilesets.

Rmk The 16 Wang tiles derived from Ammann's geometric tiles are bideterministic.

Theorem [Kari, 1991; Aanderaa-Lewis, 1974]. **DP** remains undecidable for **deterministic** tilesets.



Deterministic tilesets: a short history

[Kari-Papasoglu, 1999] builds a strong determinization of **[Robinson, 1971]**.

Theorem [Kari-Papasoglu, 1999]. There exist some **strongly deterministic** aperiodic tilesets.

[Lukkarila, 2009] introduces a strong determinization of **[Robinson, 1971] + Turing computation**.

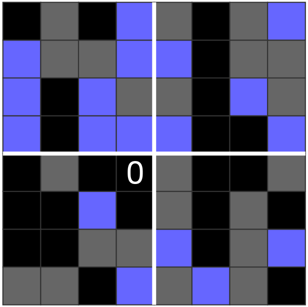
Theorem [Lukkarila, 2009]. **DP** remains undecidable for **strongly deterministic** tilesets.

2. Colorings, subshifts and (directional) soficity

Colorings of the discrete plane

Given a **finite alphabet** Σ, a **Σ-coloring** of $\mathbb{Z}^2$ is a map

$$c : \mathbb{Z}^2 \longrightarrow \Sigma$$



Topology

The set $\Sigma^{\mathbb{Z}^2}$ of Σ-colorings is endowed with the **product topology** over $\mathbb{Z}^2$ of the **discrete topology** over Σ.

Theorem. $\Sigma^{\mathbb{Z}^2}$ is a **compact** space.

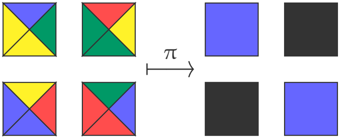
Translation. $\forall c \in \Sigma^{\mathbb{Z}^2}, \forall z, x \in \mathbb{Z}^2, \sigma_z(c)(x) = c(x - z)$

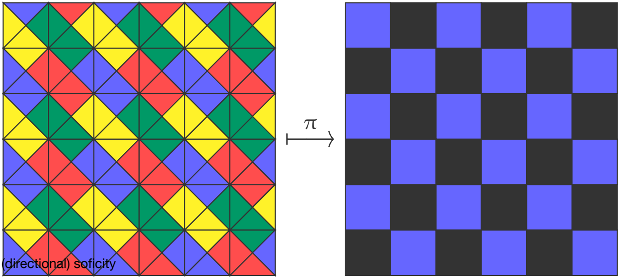
Subshift. A **subshift** $Y \subseteq \Sigma^{\mathbb{Z}^2}$ is a **close** and **translation invariant** set of colorings.

Tilings. The set of **tilings** of a tileset τ is a subshift.

2D Soficity

Soficity. A subshift $Y \subseteq \Sigma^{\mathbb{Z}^2}$ is **sofic** if it can be obtained as the **alphabetic projection** of the set of **tilings** $\mathcal{X}_\tau$ by a tileset τ: $Y = \pi(\mathcal{X}_\tau)$.





Directional soficity

Directional soficity. A subshift $Y \subseteq \Sigma^{\mathbb{Z}^2}$ is **NW/NE/SW/SE-sofic** if it can be obtained as the alphabetic projection of the set of tiling $\mathcal{X}_\tau$ by a **NW/NE/SW/SE-deterministic** tileset τ.

N.B. Those colorings are generated by **partial cellular automata** of **dimension 1**.

Motivation. Opens the doors for tools from **dimension 1**.

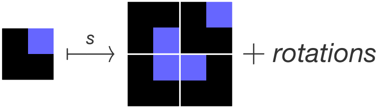
The **colorings generated by substitutions** are sofic.

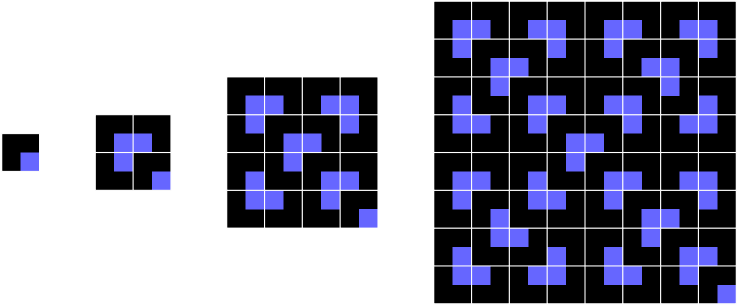
Today… What about the directional case?

3. Substitutions and soficity

Substitutions

Substitution. A (deterministic) **substitution** is a set of rules replacing letters from a finite alphabet Σ by rectangles of letters over Σ.





Limit set

What are the colorings generated by a substitution?

Limit set. Given a substitution s, we define its **limit set**:

$$\Lambda_s = \bigcap_{n \geq 0} \left\langle s^n \left(\Sigma^{\mathbb{Z}^2} \right) \right\rangle_\sigma$$

The limit set is a **subshift**.

Proposition. The limit set exactly is the set of colorings c admitting a **history**: $(c_n) \in \Sigma^{\mathbb{Z}^2}$ verifying $c_0 = c$ and $\forall n \geq 0, \exists \sigma_n$ translation, $\sigma_n \circ s(c_{n+1}) = c_n$.

Soficity of Λ_s

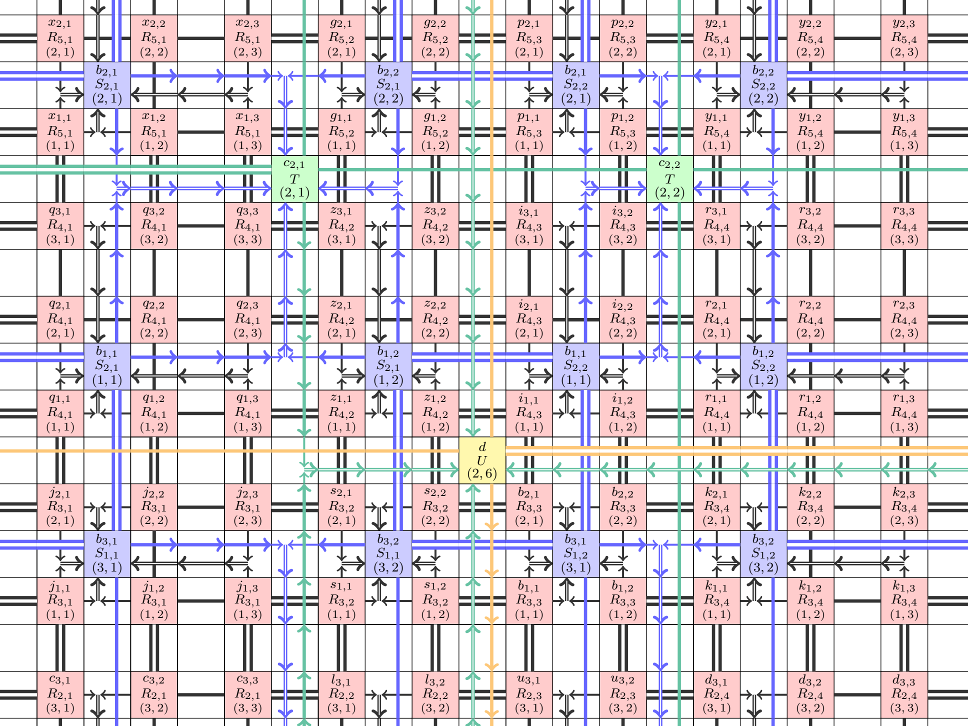
We want to force the **hierarchical structure** imposed by the **substitution** using **local rules**.

Idea. Code the **history** of a coloring into the tilings.

Theorem [Mozes, 1989]. Limit sets of (expansive) substitutions are sofic.

Seminal construction method for **[Goodman-Strauss, 1998]** and **[Fernique-O, 2010]**.

Theorem [104]. Limit sets of 2×2 substitutions are sofic.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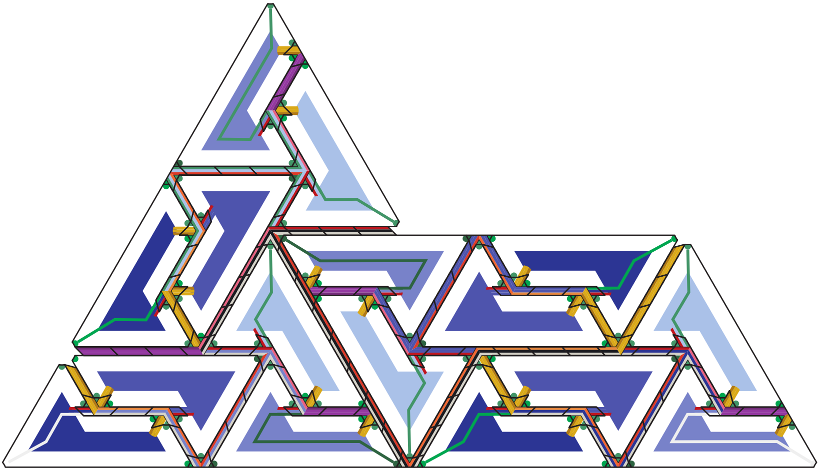
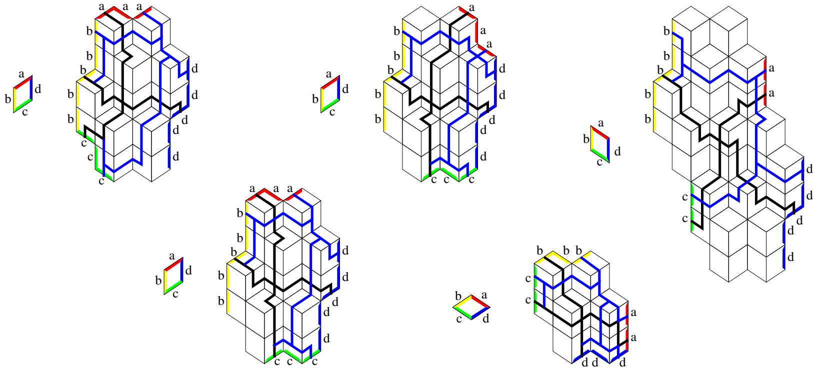
$k_{1,1}$ $R_{3,4}$ $(1,1)$
$k_{1,2}$ $R_{3,4}$ $(1,2)$
$k_{1,3}$ $R_{3,4}$ $(1,3)$
$c_{3,1}$ $R_{2,1}$ $(3,1)$
$c_{3,2}$ $R_{2,1}$ $(3,2)$
$c_{3,3}$ $R_{2,1}$ $(3,3)$
$l_{3,1}$ $R_{2,2}$ $(3,1)$
$l_{3,2}$ $R_{2,2}$ $(3,2)$
$u_{3,1}$ $R_{2,3}$ $(3,1)$
$u_{3,2}$ $R_{2,3}$ $(3,2)$
$d_{3,1}$ $R_{2,4}$ $(3,1)$
$d_{3,2}$ $R_{2,4}$ $(3,2)$
$d_{3,3}$ $R_{2,4}$ $(3,3)$

Goodman-Strauss 1998



Theorem[Goodman-Strauss 1998]. The limit set of **homothetic substitution** *(+ some hypothesis)* is sofic.

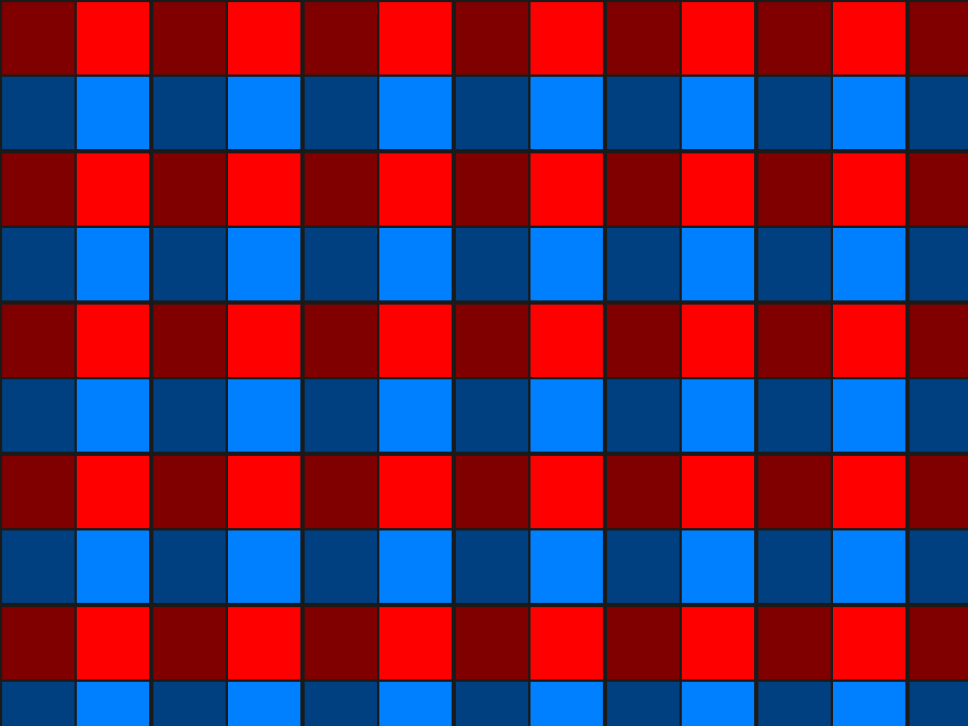
Fernique-O 2010

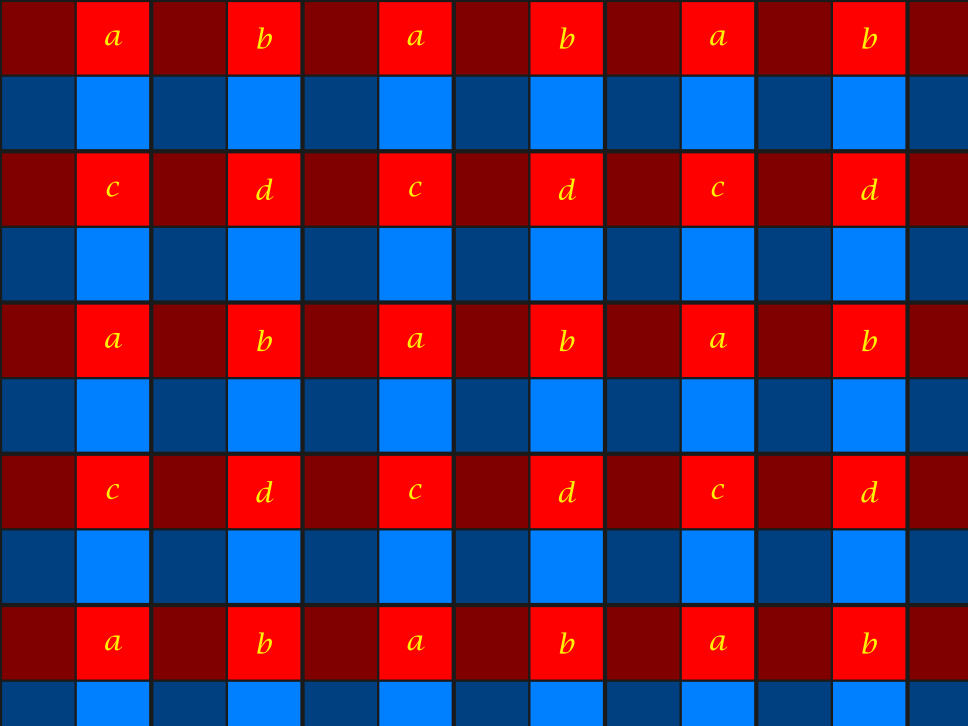


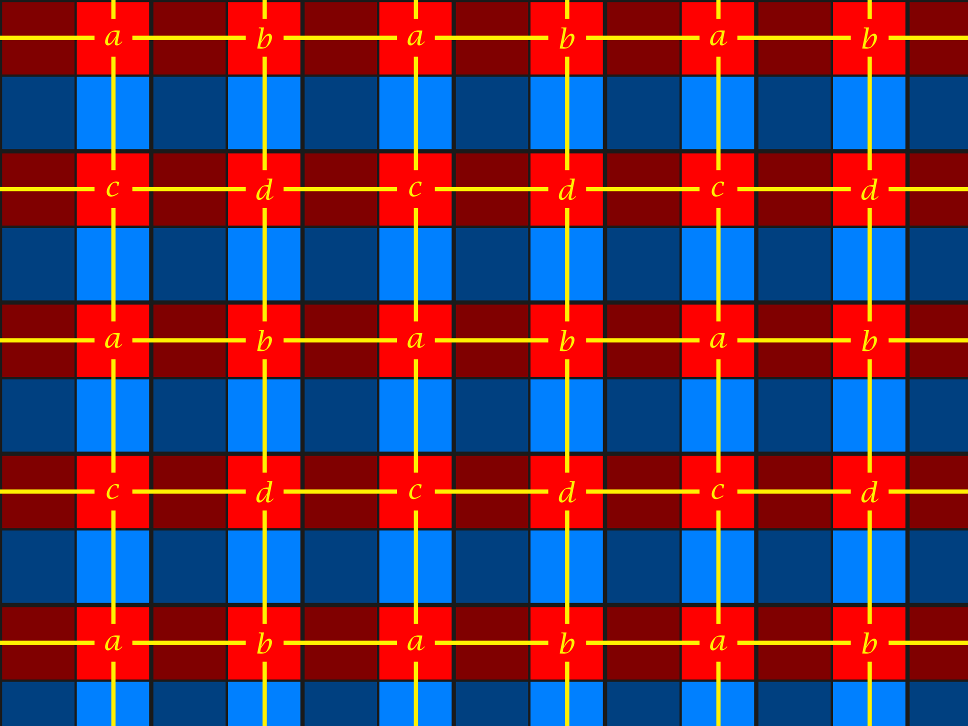
Theorem[Fernique-O 2010]. The limit set of a **combinatorial substitution** *(+ some hypothesis)* is sofic.

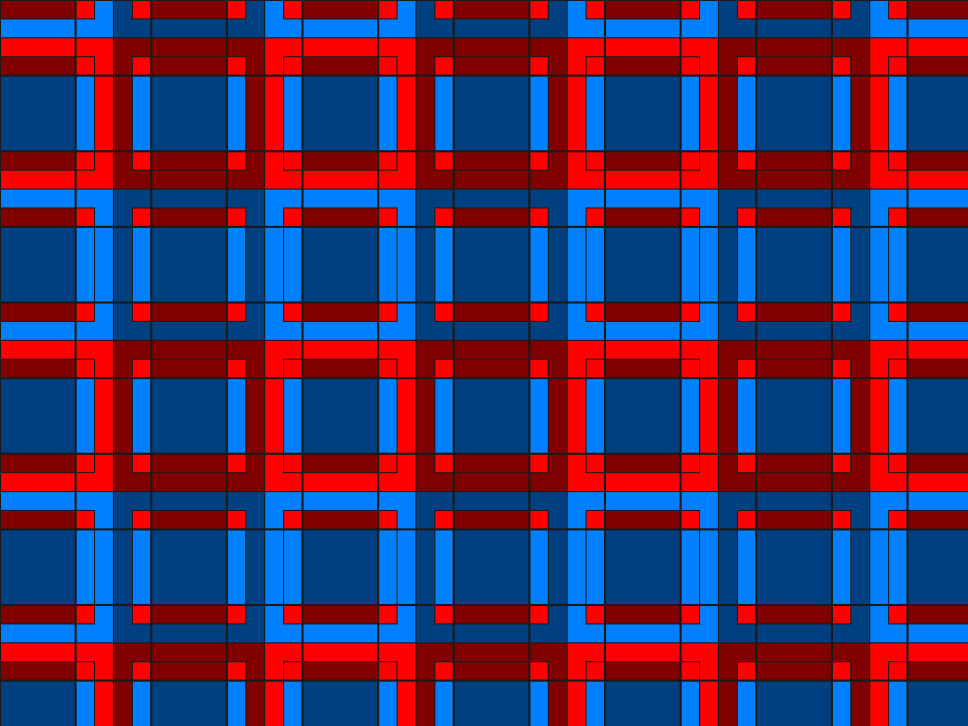
a	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*
c	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*
a	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*
c	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*
a	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*
c	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*
a	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*
c	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*
a	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*
c	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*

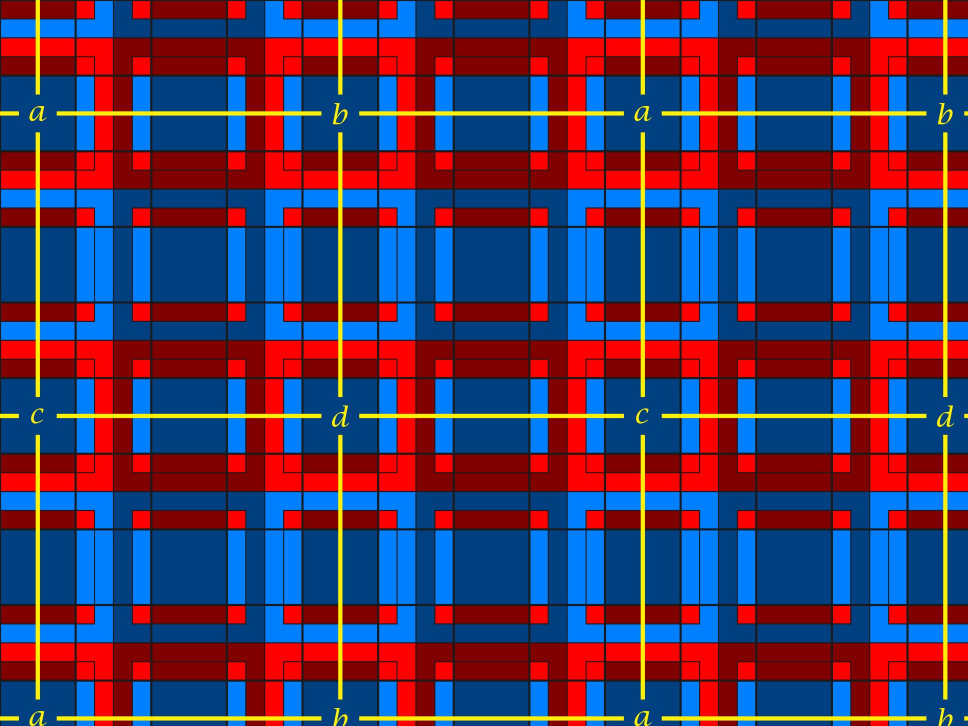
a	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*
c	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*
a	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*
c	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*
a	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*
c	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*
a	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*
c	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*
a	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*
c	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*	*d*	*c*

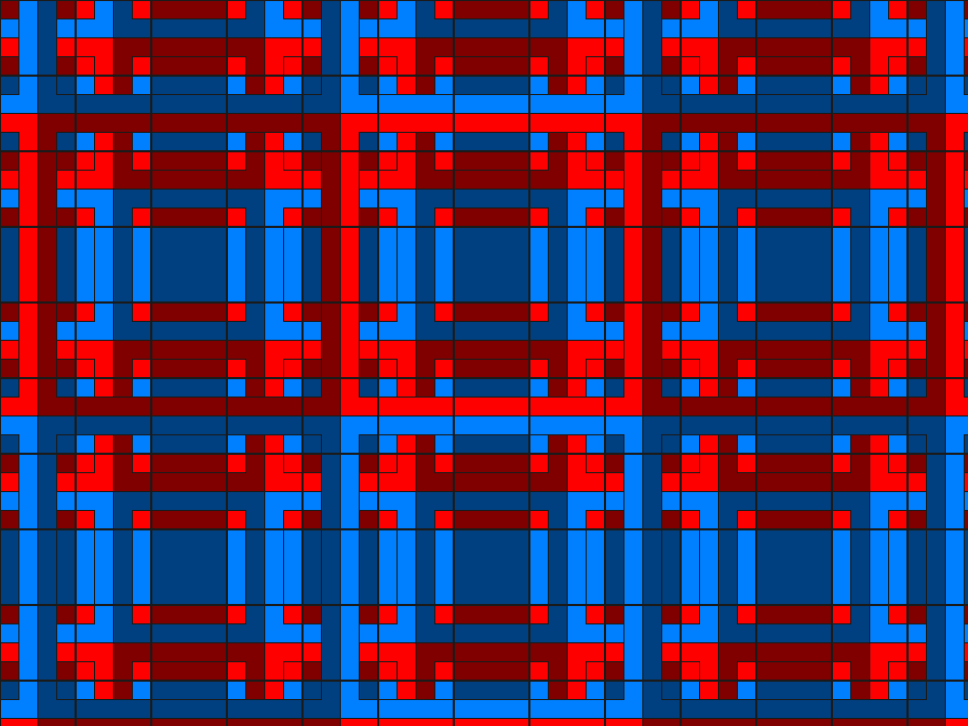


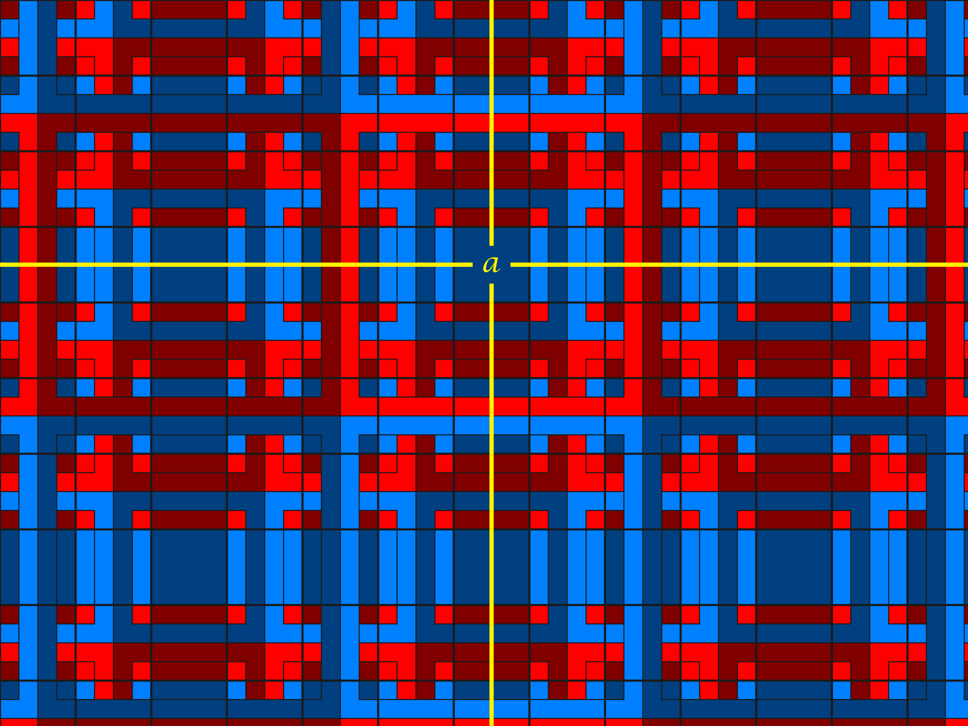

a b a b a b
c d c d c d
a b a b a b
c d c d c d
a b a b a b


a b a b a b
c d c d c d
a b a b a b
c d c d c d
a b a b a b



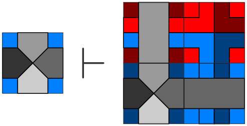

a
b
a
b
c
d
c
d
a
b
a
b




a

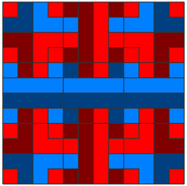
Is this self-encoding?

Iterating the coding rule one obtains 56 tiles.



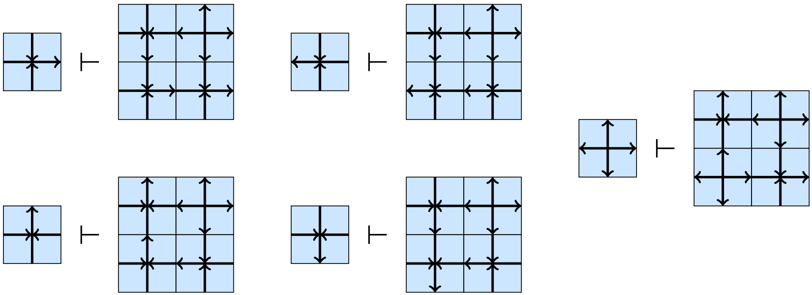
coding rule

Unfortunately, this tile set is **not self-coding**.



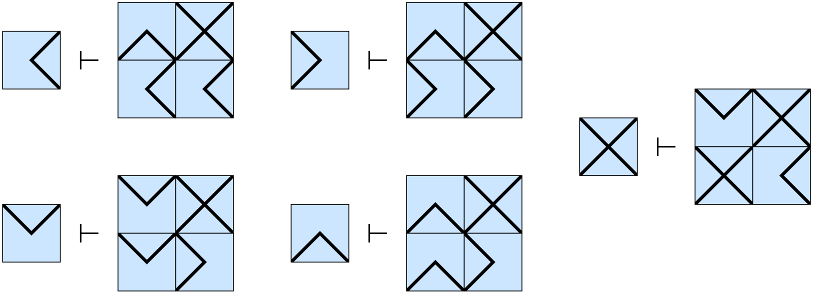
Idea Add a **synchronizing substitution** as a **third layer**.

à la Robinson

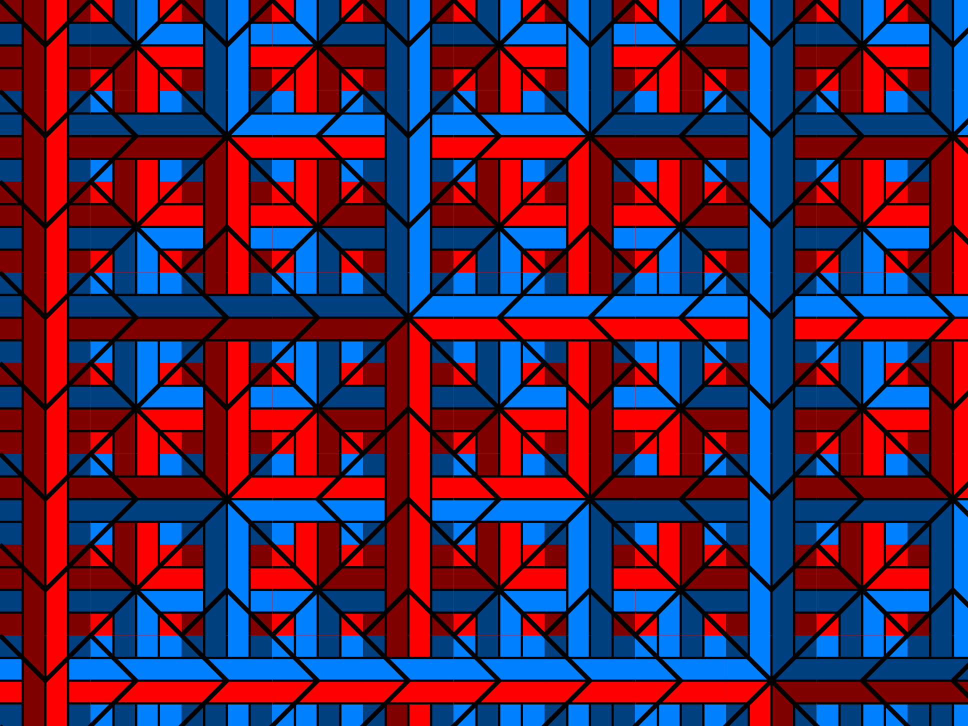


Proposition. The associated tile set of **104 tiles** admits a tiling and **codes** an **unambiguous** substitution.

à la Robinson



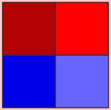
Proposition. The associated tile set of **104 tiles** admits a tiling and **codes** an **unambiguous** substitution.



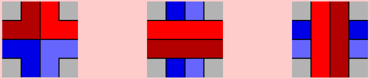
104 in brief: 3 layers

Tileset τ introduced in **[104]**.

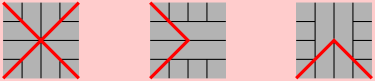
Layer 1: Parity.



Layer 2: Cables.



Layer 3.



104 in brief

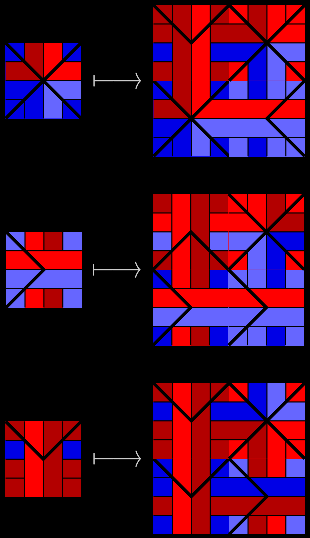
Smallest fixed point of a 2×2 **substitution scheme**.

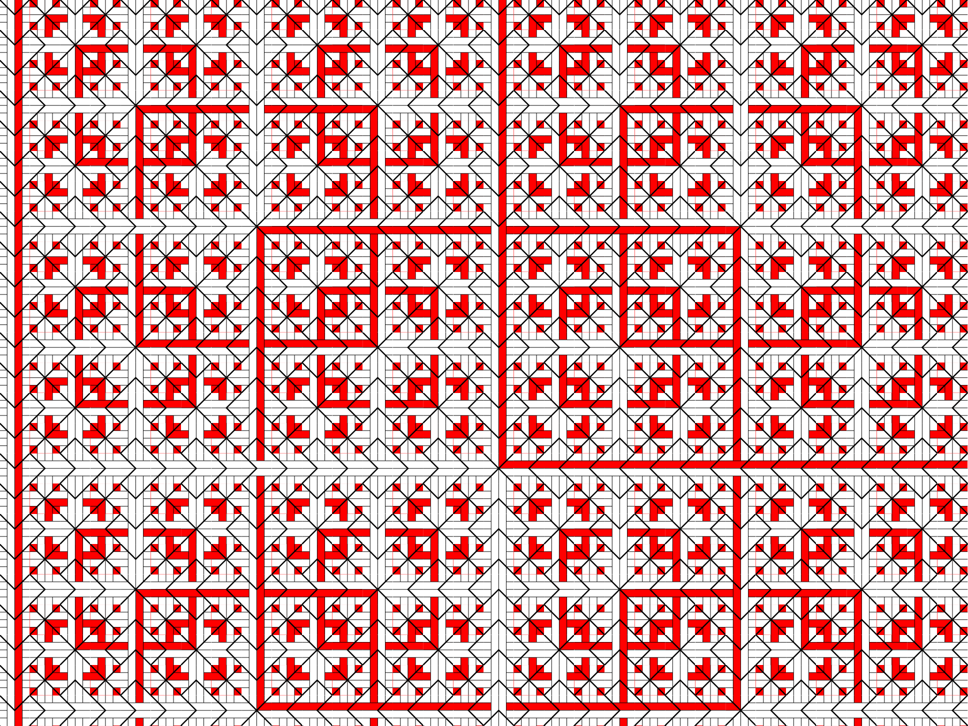
Each macro-tile **codes** a tile.

104 is **self-simulating** for a 2×2 substitution s on tiles.

The limit set Λ_s of s is **aperiodic** and the set of tilings verifies $\mathcal{X}_\tau \subset \Lambda_s$.

Theorem [104].
104 is aperiodic.





Soficity

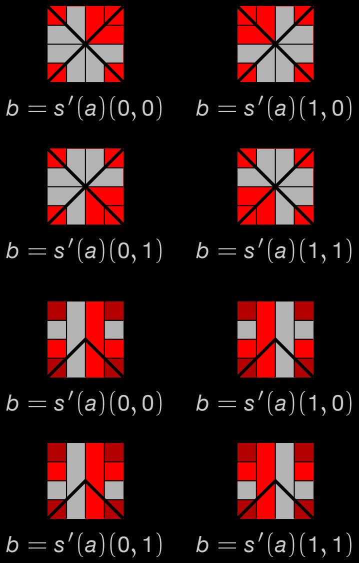
Every tilings contains an infinite **quaternary tree** hierarchical structure.

For any 2×2 substitution s' over an alphabet Σ, we enrich τ into a tileset $\tau(s')$ such that:

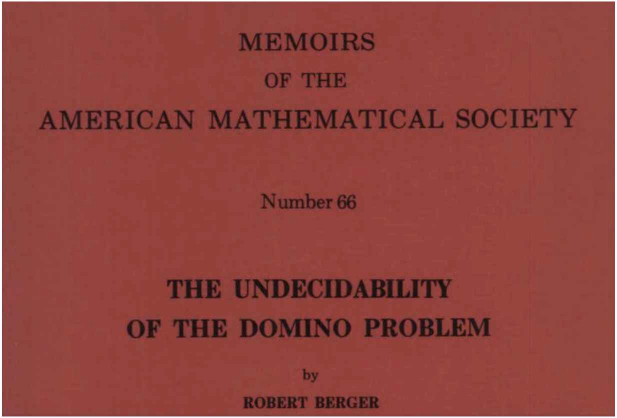
- cables of the tree structure carry letters of Σ ;
- new rules enforce the structure to code the whole history of a coloring of $\Lambda_{s'}$.

Theorem [104].

$$\pi\left(\mathcal{X}_{\tau(s')}\right) = \Lambda_{s'}$$



4. Historical interlude


MEMOIRS
OF THE
AMERICAN MATHEMATICAL SOCIETY

Number 66

THE UNDECIDABILITY OF THE DOMINO PROBLEM

by

ROBERT BERGER

"(…) In 1966 R. Berger discovered the first aperiodic tile set. It contains 20,426 Wang tiles, (…)
Berger himself managed to reduce the number of tiles to 104 and he described these in his thesis, though they were omitted from the published version (Berger [1966]). (…)" *[GrSh, p.584]*

THE UNDECIDABILITY OF THE DOMINO PROBLEM

A thesis presented

by

Robert Berger

to

The Division of Engineering and Applied Physics

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

in the subject of

Applied Mathematics

Harvard University

Cambridge, Massachusetts

July 1964

APPENDIX II

A SIMPLER SOLVABLE DOMINO SET WITH NO TORUS

The skeleton set, K, analyzed in PART 3, is a solvable domino set with no torus. Since it is designed to serve also as a base set for modeling of Turing machines, it is not surprising that simpler solvable, torus-less domino sets exist. One such set, call it Q, is specified by Tables 9-12. The first three tables show the base, skeleton, and parity prototypes of Q. Although these tables show symbols in the center of domino edges, the base, skeleton, and parity channels should be thought of as distinct. Table 12 serves the same function for Q as did Table 4 for K, namely that of specifying which products of prototypes are permitted. However, since Q is a fairly small set, it is not too cumbersome to enumerate only those dominoes which are actually used in solutions of Q, 104 in all. (No concerted attempt has been made to find the smallest solvable torus-less domino set.)

Figure 24 shows, separately, skeleton signals and parity signals in the same portion of a solution of Q. If Figure 24 is rotated one-eighth turn clockwise, its skeleton signals bear a strong resemblance to the CD-signals of K.

A person who understands the skeleton set should have no trouble convincing himself of the likelihood that all solutions of Q look line extensions of Figure 24. The following hints will help.

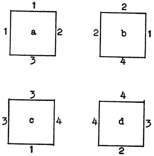
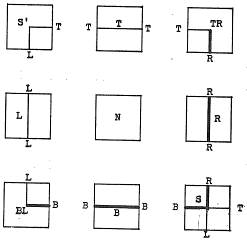


Table 9

Base Prototypes of Q



Note:

Use of same line weight for a horizontal and a (different) vertical signal introduces no ambiguity.

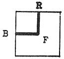


Table 10

Skeleton Prototypes of Q

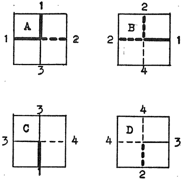
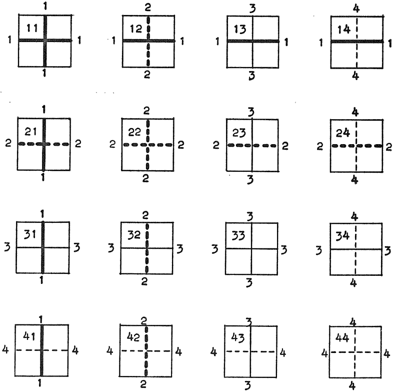


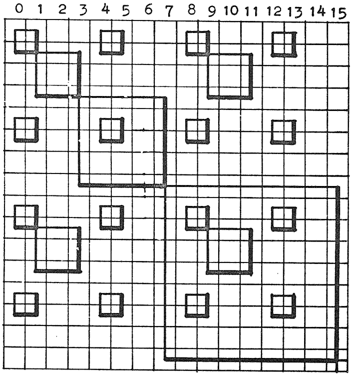
Table 11

Parity Prototypes of Q

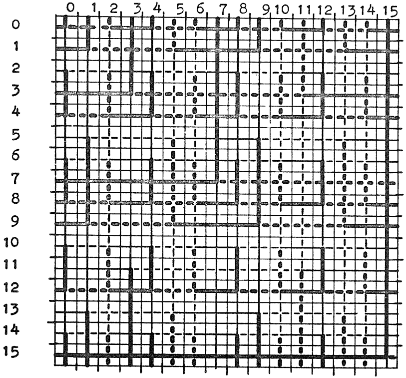


Base	Skeleton	A	B	C	D	11	12	13	14	21	22	23	24	31	32	33	34	41	42	43	44
						Parity															
a	S' S T	x																			
	L TR																				
	BL R B																				
	F N		x	x	x																
b	S' S T																				
	L TR							x		x	x	x	x			x				x	
	BL R B					x	x	x	x					x	x	x	x				
	F N					x	x	x	x					x	x	x	x	x	x		x
c	S' S T									x	x		x								
	L TR																				
	BL R B					x	x	x		x	x	x		x	x	x		x	x	x	
	F N					x	x		x	x	x			x	x		x	x	x		x
d	S' S T	x								x	x	x									
	L TR							x		x	x	x	x			x				x	
	BL R B					x x	x x	x x	x	x	x	x		x x	x x	x x	x	x	x	x	
	F N		x	x	x	x	x	x	x	x	x	■	■	x	x	x	x	x	x	■	x

Table 12 Prototype Products in Q



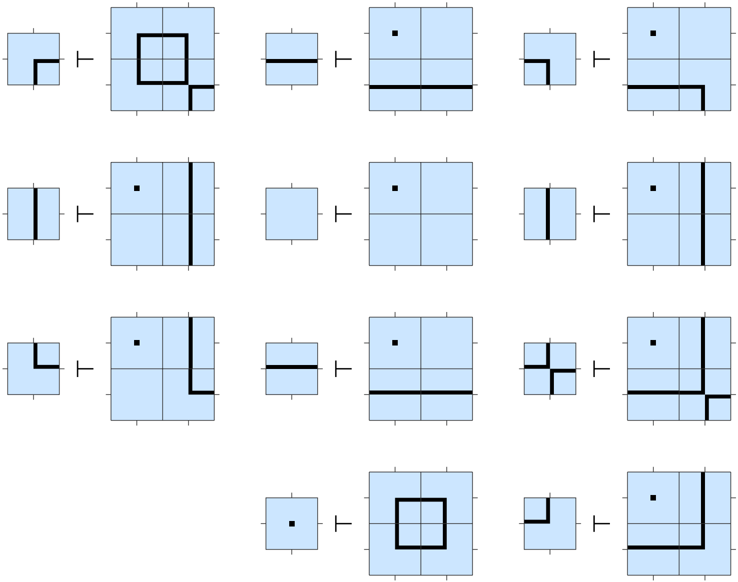
Skeleton Signals

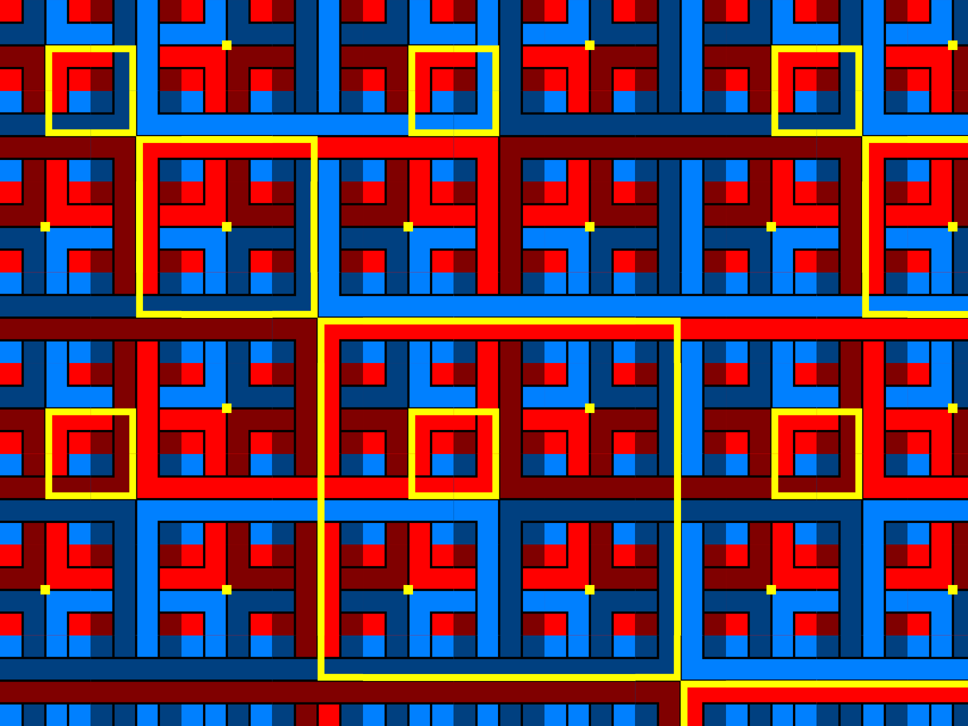


Parity Signals

Figure 24 Part of the Solution of Q

Berger’s skeleton substitution





Berger's forgotten aperiodic tile set

Proposition. The associated tile set of **103 tiles** admits a tiling and **codes** an **unambiguous** substitution.

Remark. The number of tiles **does not grow monotonically** in the number of letters of the synchronizing layer.

$$\begin{array}{lcl} \text{5 letters} & \longrightarrow & \text{104 tiles} \\ \text{11 letters} & \longrightarrow & \text{103 tiles} \end{array}$$

5. Substitutions and directional soficity

Directional soficity of Λ_s

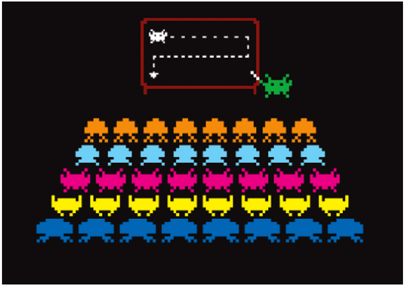
We want to force the hierarchical structure imposed by the substitution using **4-way deterministic** local rules.

In practice. Let's adapt some (rather technical) existing constructions coding the history of colorings into tilings to make them deterministic.

In the following of this talk…

Theorem. Limit sets of 2×2 substitutions are ***4-way* sofic**.

Directional soficity of Λ_s



Battle plan.

1. We determinize **104** in the **four directions** simultaneously.
2. We determinize **104 + substitutions** in **one** direction.
3. We **bi**determinize **104 + substitutions**.
4. We **strongly** determinize **104 + substitutions**.

104-way

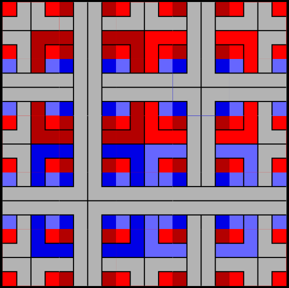
104 is not deterministic in **any** direction.

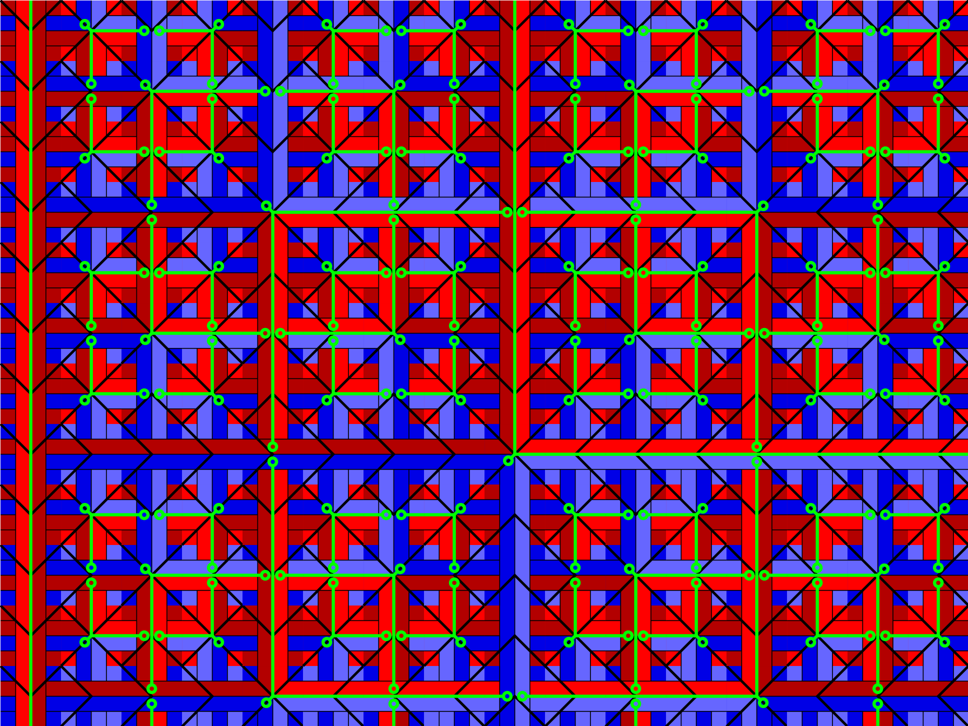
Problem. How to chose between H and V tiles in  positions?

Idea. Go & search for information on the mother tile back in the history.

Need for:

- new constraints at radius 1 for level 0 ;
- new wires (carrying H/V/X labels) to inferior levels.





104 1-way + Substitutions

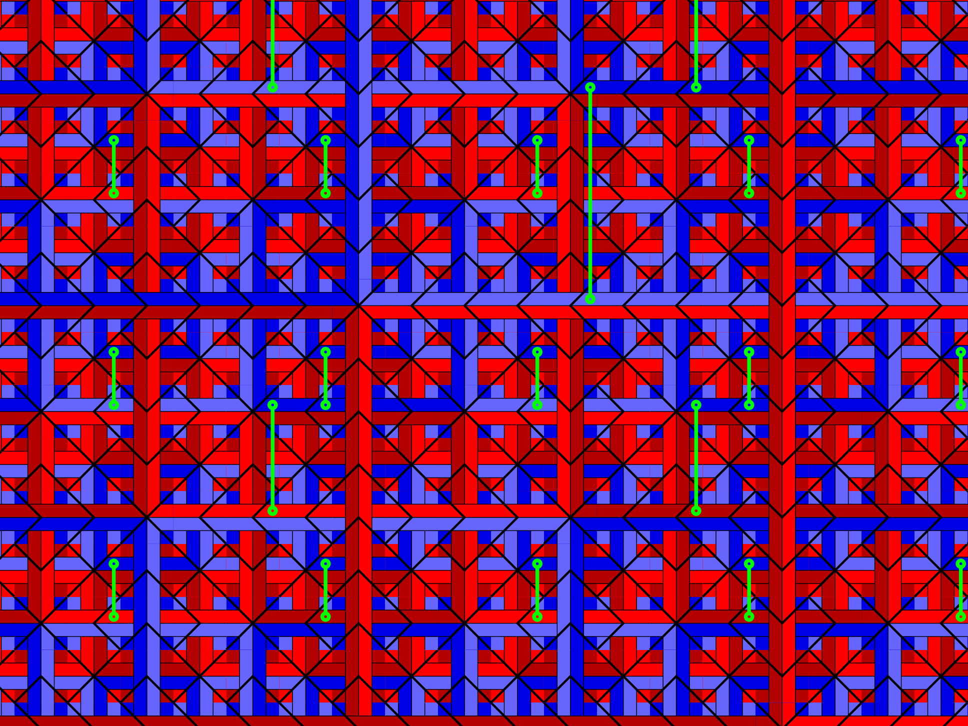
We combine the **strongly deterministic** version of 104 with the **encoding of a substitution** s' on the **quaternary tree**.

Again, the obtained tileset is not deterministic in **any** direction!

Problem. For the NE direction, we do not know how to "predict" the letter carried by cables of color  in NE position on X tiles.

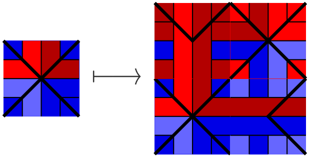
Idea. In our construction, hereditary information is **translated** in the SW direction. We could **set up some wires** to go & find it.

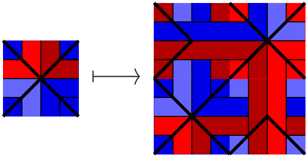
By Jove! We have already done something similar at the previous step.



104 2-way + Substitutions

We consider the cartesian product of the two **104 1-way + substitution** tilesets obtained from the following two **symmetrical substitution schemes**.

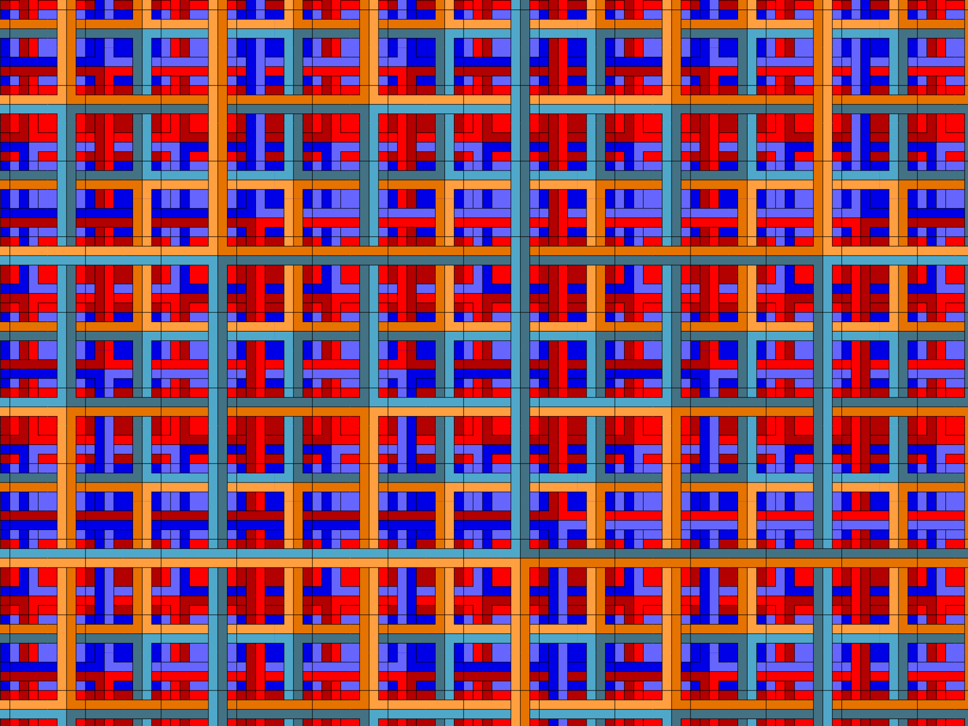


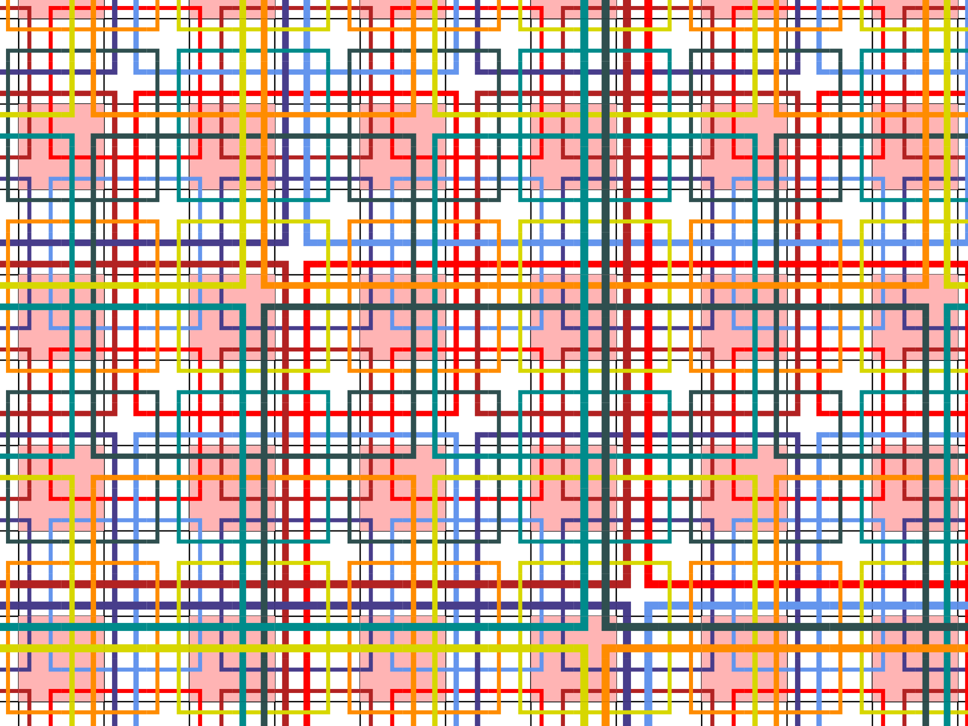


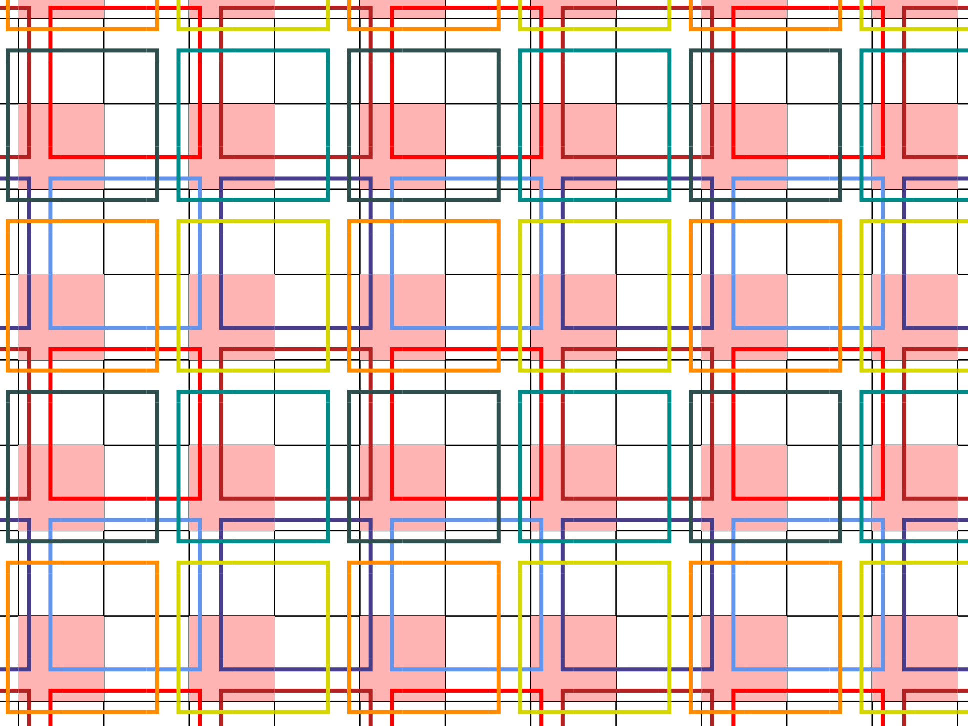
We **synchronize** the parity layer in order to **code the same coloring** on the level 0 of both components.

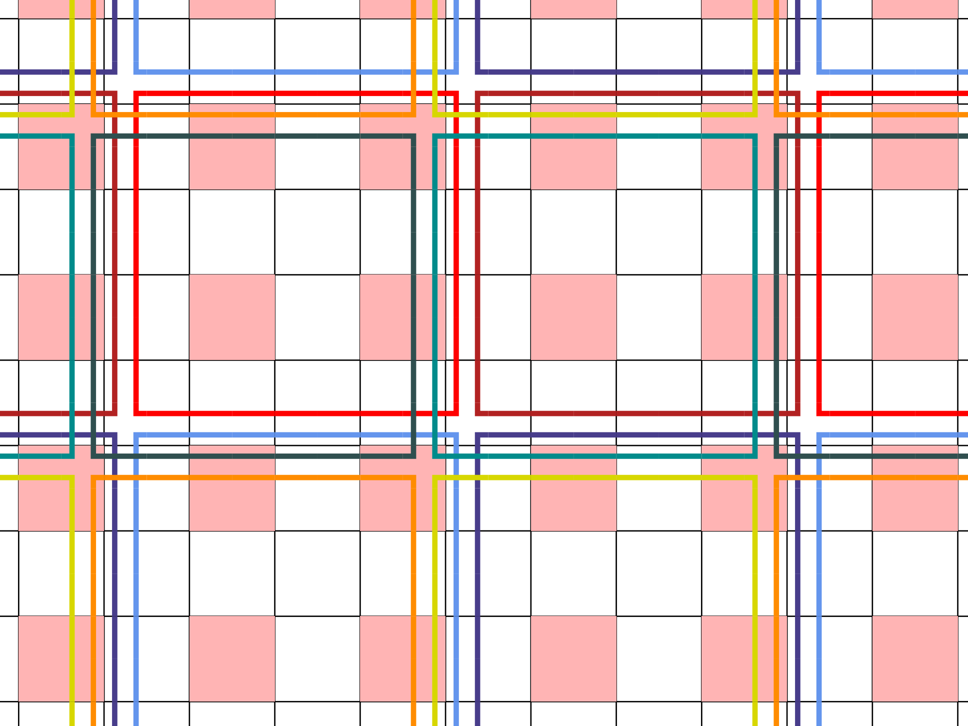
The component 1 (resp. 2) is NE-deterministic (resp. SW-deterministic).

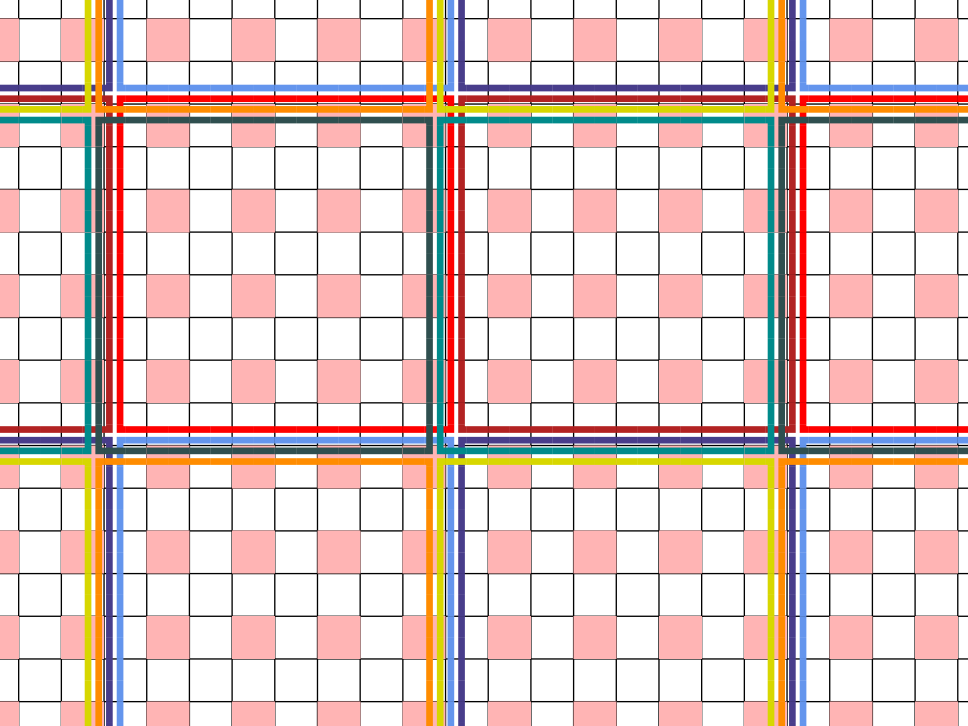
We need to synchronize the **whole history** on both components.





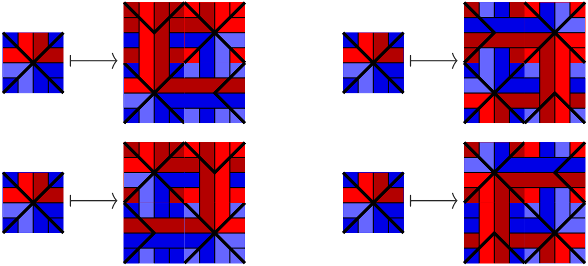




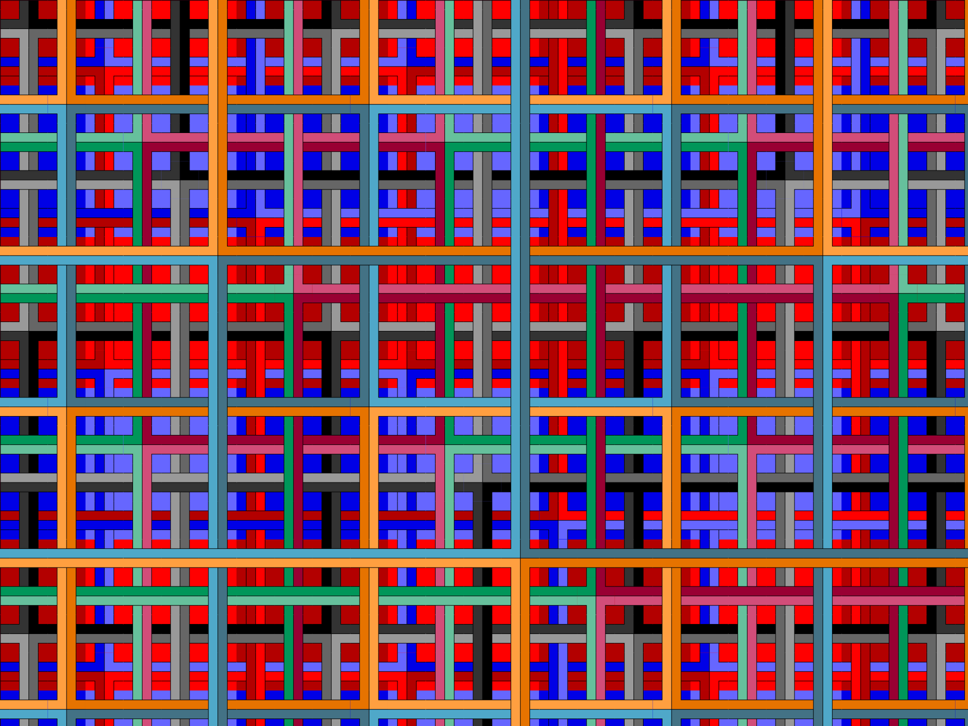


104 4-way + Substitutions

We consider this time the cartesian product of the four **104 1-way + substitution** tilesets obtained from the following four **symmetrical substitution schemes**.



Similar analysis and same solution (3×3 grouping) as before.



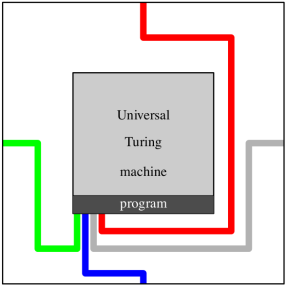
6. Going further

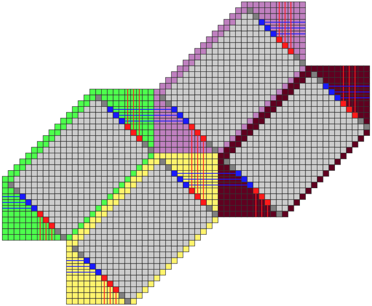
Conclusion and perspectives

We have considered the case of colorings generated by deterministic 2×2 (and $2^n \times 2^n$) substitutions.

This might be adapted for regular $n \times n$ substitutions.

General open question: What subshifts can be recognized in a deterministic way?





Bi-deterministic version of **[Durand-Romashchenko-Shen 08]**

That’s all folks!

Thank you for your attention.