Coverability as local rule

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Workshop on aperiodicity and hierarchical structures in tilings
26 September 2017
Introduction

- $\Sigma$ an alphabet, e.g. $\{□, ■\}$

- **Colorings** of groups
  - In my case, $\mathbb{Z}$ and $\mathbb{Z}^2$
Introduction

- \( \Sigma \) an alphabet, e.g. \( \{\square, \blacksquare\} \)
- **Colorings** of groups
  - In my case, \( \mathbb{Z} \) and \( \mathbb{Z}^2 \)
- **Local rules**
  - Wang tiles
  - Forbidden patterns
Introduction

- $\Sigma$ an alphabet, e.g. $\{\Box, \blacksquare\}$

- **Colorings** of groups
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- **Local rules**
  - Wang tiles
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- **Notions of regularity**
  - Periodicity
  - Repetitivity
  - Existence of frequencies
  - Entropy
• $\Sigma$ an alphabet, e.g. $\{\square, ■\}$

• **Colorings** of groups
  - In my case, $\mathbb{Z}$ and $\mathbb{Z}^2$

• **Local rules**
  - Wang tiles
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• **Notions of regularity**
  - Periodicity
  - Repetitiveness
  - Existence of frequencies
  - Entropy

• **Coverability**
"Coverable" vs. "Quasiperiodic"

**Warning**

Quasiperiodic has different meanings in different communities.

**Combinatorics on words:** quasiperiodic = coverable  
**Tilings and dynamics:** quasiperiodic = repetitive

I coined the term “coverable” to resolve this ambiguity.

But it is not standard in the literature.
Plan

1. Introduction
2. Coverability in $\mathbb{Z}$
3. Coverability in $\mathbb{Z}^2$
4. Forcing entropy with covers
5. Multi-scale coverability
1. Introduction

2. Coverability in $\mathbb{Z}$

3. Coverability in $\mathbb{Z}^2$

4. Forcing entropy with covers

5. Multi-scale coverability
Let $w, q$ be words ($q$ is finite).

**Definition**

The word $q$ is a **cover** of $w$ if $w$ is covered with copies of $q$.

- $w$ finite or infinite
- $q \neq w$
- $q$ prefix of $w$
Let $w, q$ be words ($q$ is finite).

**Definition**

The word $q$ is a *cover* of $w$ if $w$ is covered with copies of $q$.

- $w$ finite or infinite
- $q \neq w$
- $q$ prefix of $w$

**Definition**

- **Coverable** = has a cover
- **Superprimitive** = no covers
Previous work on coverability

**Text algorithms (1990’s)**
- Definition
- Detection algorithms
- Normal form

**Infinite words (2000’s)**
- Definition, questions
- Independence results
- Multi-scale case

**Characterization of covers…**
- … of Sturmian words
- … of Episturmian words

**Combinatorics (2016)**
- Tools to determine covers
- Characterize periodic words…
- …and standard Sturmian words

**On \( \mathbb{Z}^2 \) (2015, 2017)**
- Knowing “trivial” covers
- Independence results
- Multi-scale case
Normal form of coverable words

Two possibilities:
1. \( q \)
2. \( q \)

Theorem (Mouchard, 2000)
A word is \( q \)-coverable iff it is a concatenation of \( q \)-antiborders, starting with \( q \).

Border: prefix + suffix
Antiborder: right complement of a border
Normal form of coverable words

Two possibilities:

1. \(q\) \(\{\)
   \[
   \begin{array}{ccccccc}
   & & & & & & \\
   & & & & & & \\
   & & & & & & \\
   & & & & & & \\
   \end{array}
   \]
   \(q\) \(\}\)

2. \(q\) \(\{\)
   \[
   \begin{array}{ccccccc}
   & & & & & & \\
   & & & & & & \\
   & & & & & & \\
   & & & & & & \\
   \end{array}
   \]
   \(q\) \(\}\)

Theorem (Mouchard, 2000)

A word is \(q\)-coverable iff it is a concatenation of \(q\)-antiborders, starting with \(q\).
Two possibilities:

1. \( q \) \( \text{border} \) \( \text{antiborder} \)

2. \( q \) \( \text{antiborder} \) 

**Theorem (Mouchard, 2000)**

A word is \( q \)-coverable iff it is a concatenation of \( q \)-antiborders, starting with \( q \).
Normal form of coverable words

Two possibilities:

1. \( q \)
   \[ \square \square \square \square \square \quad \square \square \square \square \square \]

2. \( q \)
   \[ \square \square \square \square \square \quad \square \square \square \square \square \]
      \[ \text{border} \]
   \[ \square \square \square \square \square \quad \square \square \square \square \square \]
      \[ \text{antiborder} \]

Theorem (Mouchard, 2000)

A word is \( q \)-coverable iff it is a concatenation of \( q \)-antiborders, starting with \( q \).

- **Border**: prefix + suffix
- **Antiborder**: right complement of a border
Substitutions from covers

Fix a word $q$, say with $n$ antiborders.

Definition

$\mu_q(i)$ is the $i^{th}$ antiborder of $q$
(by decreasing size)

Example

$q = □■□■□■□□□$

$\mu_q(0) = □■□■□□□□$

$\mu_q(1) = ■□□□□□□$

$\mu_q(2) =$

■□□□□□
Substitutions from covers

Fix a word $q$, say with $n$ antiborders.

**Definition**

$q(i)$ is the $i^{\text{th}}$ antiborder of $q$ (by decreasing size)

Now view $q$ as a substitution $\{0, \ldots, n-1\}^* \rightarrow \Sigma^*$.

**Example**

\[
q = \begin{array}{c}
\text{□■□■□■□■□□}
\end{array}
\]

\[
q(0) = \begin{array}{c}
\text{□■□■□■□■□□}
\end{array}
\]

\[
q(1) = \begin{array}{c}
\text{■□■■□■□□}
\end{array}
\]

\[
q(2) = \begin{array}{c}
\text{■■□■□□}
\end{array}
\]
Fix a word \( q \), say with \( n \) antiborders.

**Definition**

\[ \mu_q(i) \] is the \( i \)\(^{\text{th}} \) antiborder of \( q \)
(by decreasing size)

Now view \( \mu_q \) as a substitution
\( \{0, \ldots, n-1\}^* \to \Sigma^* \).

**Example**

\[ q = \text{□■□■■□■□} \]
\[ \mu_q(0) = \text{□■□■■□■□} \]
\[ \mu_q(1) = \text{■□■■□■□} \]
\[ \mu_q(2) = \text{■■□■□} \]

**Theorem** (Mouchard, 2000)

A word \( w \) is \( q \)-coverable iff
\[ \exists u \text{ such that } w = \mu_q(0 \cdot u) \]
Remark

For most $q$, $\mu_q$ preserves interesting properties

For instance,

- Non-repetitivity
- Positive entropy
- Divergence of frequencies

Thus we can create

irregular coverable words
Remark
For most $q$, $\mu_q$ preserves interesting properties

For instance,
- Non-repetitivity
- Positive entropy
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Thus we can create
**irregular coverable words**

[Marcus, Monteil 2006]
If \( q = \square \), there is only one \( q \)-coverable word: \( \square^\mathbb{Z} \).
If $q =  \square$, there is only one $q$-coverable word: $\square^\mathbb{Z}$.

**Theorem**

*If $\mu_q$ is not injective (on infinite words) then $\forall u, \mu_q(u) = q^\mathbb{Z}$.***

We have a **dichotomy**:

- either there exist irregular $q$-coverable words,
- or all $q$-coverable words are periodic.

Besides, injectivity of $\mu_q$ is equivalent to an easy combinatorial condition on $q$. (More on this later.)
Plan

1. Introduction
2. Coverability in \( \mathbb{Z} \)
3. Coverability in \( \mathbb{Z}^2 \)
4. Forcing entropy with covers
5. Multi-scale coverability
A configuration is a coloring of $\mathbb{Z}^2$. A block is a coloring of a finite rectangle.

**Definition**

Let $q$ be a block. A configuration $w$ is $q$-coverable if it is covered with copies of $q$. 
Notions of regularity

Definitions

- **Block complexity**
  \[ P_w(m, n) = \# \text{ blocs } (m, n) \text{ in } w \]

- **Entropy**
  \[ \text{Ent}(w) = \lim \log(P_w(n, n))/n^2 \]

- **Block frequencies**
  \[ f_w(b) = \text{average number of } b-\text{occurrences per cell} \]

- **Repetitiveness**
  Each block occurs \(\infty\) often with bounded gaps

Plan

Show that **coverability is independent of these**...
... but we have **no more normal form!**
Ruling out “trivial” covers

The cover □ only allows □^Z^2.
Ruling out “trivial” covers

The cover □ only allows □^\mathbb{Z}^2.

Theorem (Richomme and G.)

Let \( q \) be a block. There exists an aperiodic, \( q \)-coverable configuration iff the primitive root of \( q \) has a nonempty border.
Ruling out “trivial” covers

The cover □ only allows □_{\mathbb{Z}^2}.

Theorem (Richomme and G.)

Let \( q \) be a block. There exists an aperiodic, \( q \)-coverable configuration iff the primitive root of \( q \) has a nonempty border.

Ideas of the proof

1. If the root has no border, all overlaps are multiples of the root
2. Build tiles from \( q \) and freely tile the plane

Border
Block in two opposite corners

Primitive root
Unique minimal \( \nu \) such that \( u = \nu^{m \times n} \)
Ruling out “trivial” covers

The cover □ only allows □ \( \mathbb{Z}^2 \).

**Theorem** (Richomme and G.)

Let \( q \) be a block.
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Ruling out “trivial” covers

The cover □ only allows □\(\mathbb{Z}^2\).

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2. Build tiles from \(q\) and freely tile the plane
The tiles

Coverability as local rule
Coverable configurations

Remark

\( f(w) \) is defined for \( w \in \{a, b, c, d\}^\mathbb{Z}^2 \)
only if \( w \) satisfies some local rules
(More about this on the next slide)

Proposition (Richomme and G.)

\( \forall w, f(w) \) is \( q \)-coverable if it exists
Moreover, \( f \) preserves
- periodicity
- repetitivity
- existence of frequencies
Local rules and entropy

Remark

There are configurations
- aperiodic
- non-repetitive
- without frequencies
and matching these rules.

Remark

The rules force zero entropy.

Which covers allow positive entropy?
1. Introduction
2. Coverability in $\mathbb{Z}$
3. Coverability in $\mathbb{Z}^2$
4. Forcing entropy with covers
5. Multi-scale coverability
Fix some block \( q \).

**What we want**

Conditions on \( q \) implying

1. zero entropy for all configurations
2. positive entropy for some configurations which are \( q \)-coverable.

**Tool:** interchangeable pairs
Fix some block $q$.

**What we want**

Conditions on $q$ implying

1. zero entropy for all configurations
2. positive entropy for some configurations which are $q$-coverable.

**Tool:** interchangeable pairs

**Definition**

An interchangeable pair is a pair of $q$-coverable patterns with the same shape. (Not always rectangles.)

**Definition**

An interchangeable pair is **valid** if its shape can tile the plane.
Fix a cover $q$ and let $h = \max\{\text{Ent}(w), \ w \text{ is } q-\text{coverable}\}$.

**Theorem**

- If there is a *valid* pair for $q$, then $h > 0$.
- If there is no valid pair for $q$, then $h = 0$.

Let $u$ be a configuration with positive entropy. Consider $\mu(u)$.

- Let $v$ be an $n \times n$-square in a $q$-coverable configuration $w$.
- Let $\bar{v}$ be the smallest $q$-coverable pattern in $w$ containing $v$.
- Then $v$ is determined by the shape of $\bar{v}$ and coordinates.
- We have less than $|\Sigma|^{4n|q|} \times n^2$ possibilities.
Lemma 1
Any cover with full-width or full-height border allows positive entropy.

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Coverability as local rule
Lemma 1
Any cover with full-width or full-height border allows positive entropy.

Lemma 2
Any cover with one of these shapes allows positive entropy.
Covers allowing positive entropy

Lemma 1
Any cover with full-width or full-height border allows positive entropy.

Lemma 2
Any cover with one of these shapes allows positive entropy.

\[
\begin{array}{c|c}
    a & b \\
    b & a \\
\end{array}
\]

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Coverability as local rule
A sufficient condition for zero entropy

Theorem (Richomme and G.)
If $q$ has a corner without borders, then any $q$-coverable configuration has zero entropy.

Example
Suppose there are no overlaps like:

```
+---+
|   |
+---+
```

What occurrences are covering the $\alpha$'s?
A sufficient condition for zero entropy

**Theorem** (Richomme and G.)
If \( q \) has a corner without borders, then any \( q \)-coverable configuration has zero entropy.

**Example**
Suppose there are no overlaps like:

There are three cases.
A sufficient condition for zero entropy

**Theorem** (Richomme and G.)

If $q$ has a corner without borders, then any $q$-coverable configuration has zero entropy.

**Example**

Suppose there are no overlaps like:

What occurrences are covering the $\alpha$’s?
Theorem (Richomme and G.)

If $q$ has a corner without borders, then any $q$-coverable configuration has zero entropy.

Example

Suppose there are no overlaps like:

The occurrence covering $\alpha$ is unique in all cases.
A sufficient condition for zero entropy

**Theorem** (Richomme and G.)
If $q$ has a corner without borders, then any $q$-coverable configuration has zero entropy.

**Example**
Suppose there are no overlaps like:

The occurrence covering $\alpha$ is unique in all cases.

$\Rightarrow$ the shape of a $q$-coverable pattern determines the pattern itself.
A sufficient condition for zero entropy

Theorem (Richomme and G.)
If \( q \) has a corner without borders, then any \( q \)-coverable configuration has zero entropy.

Example
Suppose there are no overlaps like:

The occurrence covering \( \alpha \) is unique in all cases.

\[ \implies \text{the shape of a } q\text{-coverable pattern determines the pattern itself} \]
\[ \implies \text{no interchangeable pairs} \]
Lemma

Suppose \( q \) has no pairs of borders \((a, b)\) such that

\[
\begin{align*}
    w(a) + w(b) & \geq w(q) \quad \text{or} \\
    h(a) + h(b) & \geq h(q)
\end{align*}
\]

then any \( q \)-coverable configuration has zero entropy.

Not

\[
\begin{array}{c}
    a \\
    b \\
    a
\end{array}
\]
Lemma

Suppose \( q \) has no pairs of borders \((a, b)\) such that

\[
\begin{align*}
w(a) + w(b) & \geq w(q) \\
h(a) + h(b) & \geq h(q)
\end{align*}
\]

then any \( q \)-coverable configuration has zero entropy.

Proof

Same ideas as previous proof, but more cases to check.
Recap about entropy

We have

\[
\begin{array}{c|c}
  a & b \\
  b & a \\
\end{array}
\quad \Rightarrow \quad \text{positive entropy}
\]

Not

\[
\begin{array}{c|c}
  a & b \\
  b & a \\
\end{array}
\quad \Rightarrow \quad \text{zero entropy}
\]

Not quite an “if and only if”, but we’re getting close.

Remark

The duality in 1D does not apply in 2D: the cover have aperiodic configurations, but all with zero entropy.
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Definition

A word, configuration is **multi-scale coverable** if it has infinitely many covers (growing in all directions).

---

**Repetitivity**

**Zero Entropy**

**Existence of frequencies**

**Good notion of regularity**

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[Marcus, Monteil 2006]
Definition

A \{\text{word, configuration}\} is \textbf{multi-scale coverable} if it has infinitely many covers (growing in all directions).

- **Multi-scale** implies:
  - Repetitivity
  - Zero Entropy
  - Existence of frequencies

[Marcus, Monteil 2006]
**Multi-scale coverability**

**Definition**
A word, configuration \( g \) is **multi-scale coverable** if it has infinitely many covers (growing in all directions).

- **Multi-scale** implies:
  - Repetitivity
  - Zero Entropy
  - Existence of frequencies

- **Good notion of regularity**

[Marcus, Monteil 2006]
**Reminder** (Marcus and Monteil)

Any **1D** multi-scale word has
- Repetitivity
- Zero entropy
- Existing frequencies

**Question**

What about multi-scale configurations?

**Theorem** (Richomme and G.)

Any multi-scale configuration has
- Zero entropy
- Existing frequencies

**Proof sketch**

1. Direct adaptation of 1D proof
2. Lots of calculations
**Multi-scale coverability in 2D**

**Reminder** *(Marcus and Monteil)*

Any 1D multi-scale word has
- Repetitivity
- Zero entropy
- Existing frequencies

**Question**

What about multi-scale configurations?

**Theorem** *(Richomme and G.)*

Any multi-scale configuration has
1. Zero entropy
2. Existing frequencies
Multi-scale coverability in 2D

Reminder (Marcus and Monteil)

Any **1D** multi-scale word has
- Repetitivity
- Zero entropy
- Existing frequencies

Question

What about multi-scale configurations?

Theorem (Richomme and G.)

Any multi-scale configuration has
1. Zero entropy
2. Existing frequencies

Proof sketch

1. Direct adaptation of 1D proof
2. Lots of calculations
Repetitivity of multi-scale configurations

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Coverability as local rule
Repetitivity of multi-scale configurations
**Conclusion**

- **Coverability** comes from the study of finite and $\mathbb{Z}$-words
- On $\mathbb{Z}^2$: characterization of *trivial* covers
- Ongoing characterization of *covers forcing zero entropy*
- **Multi-scale coverability** is a good notion of regularity

Many possible extensions:
- as a local rule
- as a notion of regularity

Thank you for your attention!