Random walks on graphs induced by aperiodic tilings

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Seminal paper, Pólya, 1921

- $\mathcal{S} = \{\pm\varepsilon_i : i = 1, \ldots, i = d\} \subset \mathbb{Z}^d$
- $\{\xi_k\}_{k\geq 1}$ a sequence of independent and identically distributed r.v. with common distribution $\mathbb{P}(\xi_1 = \pm\varepsilon_i) = 1/2d$
- $S_n = x + \sum_{k=1}^{n} \xi_k$

Question : What can we say about the probability

$$p = \mathbb{P}(S_n = x \text{ for infinitely many } n)?$$

Answer (partial) :

$$\text{if} \quad d = 1, 2 \quad p = 1 \quad \text{and} \quad p = 0 \quad \text{otherwise}$$

Remark : By construction, the random walk is homogeneous in the sense that

$$\forall y \in \mathbb{Z}^d : \mathcal{L}(y + S_{n+1} | y + S_n) = \mathcal{L}(S_{n+1} | S_n)$$

Motivation : break the symmetries of the lattice

(Undirected) graphs and random walks

An undirected graph is a couple (V, E) where

1. V is a countable set of *nodes*;
2. E is a set of *unordered pairs* $\{x, y\}$ with $(x, y) \in V \times V$.

The *degree*, deg x, of a nodes $x \in V$ is card $\{y \in V : \{x, y\} \in E\}$.

Generally, it is assumed that the graph (V, E) is *connected*, *locally finite* and *uniformly bounded* which corresponds respectively to

1. for all $x, y \in V$ there exists a path $(z_0 = x, z_1, \dots, z_n = y)$ of nodes s.t. $\{z_i, z_{i+1}\} \in E$ for $i = 0, \dots, n-1$,
2. $\deg x < \infty$ for all $x \in V$,
3. $\sup_{x \in V} \deg x < \infty$.

Random walks on graphs

A random walk on a graph (V, E) is a Markov chain $(M_n)_{n\geq 0}$ taking values in V whose transition kernel is adapted to the graph structure, *i.e.* for all $x, y \in V$:

$$P(x,y) > 0 \iff \{x,y\} \in E.$$

A random walk on a graph is said to be *simple* if the transition kernel is given for all $x, y \in V$ by

$$P(x,y) = \begin{cases} \frac{1}{\deg\ (x)} & \text{if } \{x,y\} \in V, \\ 0 & \text{otherwise.} \end{cases}$$

Example 1 : homogeneous spaces

Let us denote by

- $\mathfrak{S}(V)$ the group of permutations of V,
- $\mathrm{Aut}(P) = \{g \in \mathfrak{S}(V) : \forall x, y \in \mathbb{G}^0, P(g \cdot x, g \cdot y) = P(x, y)\}$ the group of automorphism of P.

The Markov chain on V with transition kernel P is said *homogeneous* if the action of $\mathrm{Aut}(P) \curvearrowright V$ is transitive.

Homogeneous MC $\leftrightsquigarrow$ RW on the group of automorphisms

Up to symmetries, they are processes with independent and stationnary increments

Example 2 : triangulations of planar surface

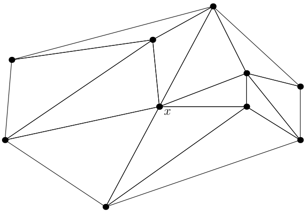

FIGURE: Patch of a triangulation

Theorem (Dodziuk, 1984)

If for some $d \geq 0$

$$7 \leq \deg(x) \leq d$$

for all $x \in V$, then the simple random walk is transient.

Example 3 : circle packing of the plane

FIGURE: Circle packing

Theorem

If $\deg(x) \leq 6$ for all but finitely many points x, then the simple random walk is recurrent.

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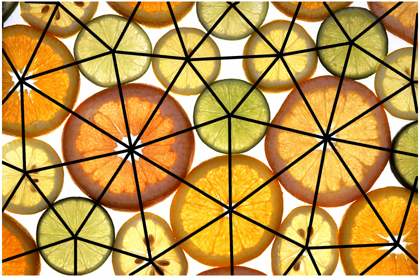

FIGURE: Circle packing

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If $\deg(x) \leq 6$ *for all but finitely many points* x*, then the simple random walk is recurrent.*

Third Penrose tiling

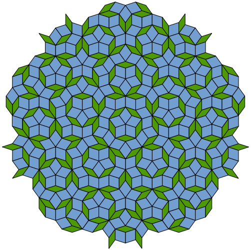


FIGURE: Third Penrose tiling

Two prototiles :

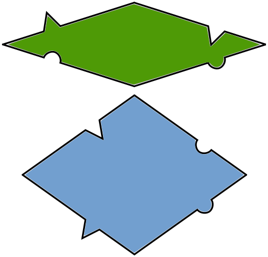
- a fat rhombus (angles of measure $2\pi/5$ and $3\pi/5$),
- a thin rhombus (angles of measure $\pi/5$ and $4\pi/5$).

together with matching rules

$$\Downarrow$$

Aperiodic tiling of $\mathbb{R}^2$

Third Penrose tiling



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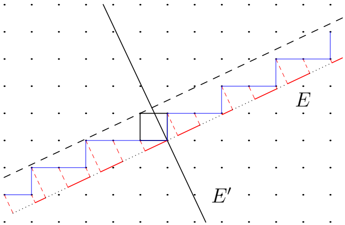

FIGURE: Linear tiling

Let

- $\mathbb{Z}^2 \subset \mathbb{R}^2$,
- E a vector line with $E \cap \mathbb{Z}^2 = \{0\}$,
- E' the orthogonal supplement of E,
- K the unit cube of $\mathbb{Z}^2$
- the strip $\mathscr{K}_t = K + E + t$, $t \in E'$.

Orthogonal projection of edges entirely contained in the strip

$\Downarrow$

Aperiodic tiling of E with short and long segments

Main goal : tile a d-dimensional vector subspace $E \subset \mathbb{R}^N$

1. $\mathbb{Z}^N \subset \mathbb{R}^N$, $\mathcal{S} = \{\pm\varepsilon_i : 1 \leq i \leq N\}$,
2. $\mathbb{R}^N = E \oplus E'$ avec dim $E = d$, π and π' the corresponding projectors,
3. $K = \{\sum_{i=1}^N \lambda_i \varepsilon_i : \lambda_i \in [0,1], 1 \leq i \leq N\}$,
4. $M_d = \left\{ I = \{i_1, \cdots, i_d\} \subset \{1, \cdots, N\} \right\}$,
5. $K_I = \{\sum_{\ell=1}^d \lambda_{i_\ell} \varepsilon_{i_\ell}, \lambda_{i_\ell} \in [0,1], i_\ell \in I\}$,
6. $K_{I^\complement}$ denotes the supplementary facet of K_I in the unit cube K,
7. $D_I = \pi(K_I)$ et $D'_I = \pi'(K_{I^\complement})$,
8. **NDEG assumption :** d-dimensional facets of $K \cong D_I$ & $(N-d)$-dimensional facets of $K \cong D'_I$.

Theorem (Oguey, Duneau and Katz)

For $t \in E'$, we set

$$\mathcal{T}_t = \left\{ x + D_I : x = \pi(\xi), \xi \in \mathbb{Z}^N : \pi'(\xi + K_I) \subset \pi'(K) + t, I \in M_d \right\}$$

□

Theorem (Oguey, Duneau, Katz, 1988)

The $\mathcal{T}_t$ is a tiling of $E \cong \mathbb{R}^d$ for all $t \in E'$ generic (or non ambiguous).

Remarks :

1. Meaning : there exists a unique d-dimensional faceted manifold entirely contained in the strip $\mathscr{K}_t$ (for any generic $t \in E'$) ;
2. group of translations of the tiling : $E \cap \mathbb{Z}^N$.

Examples

Example 1, the third Penrose tiling :

In $\mathbb{R}^5$, set $E = \mathrm{Span}(v_1, v_2)$ with

$$\begin{aligned} v_1 &= (1, \cos(2\pi/5), -\cos(\pi/5), -\cos(\pi/5), \cos(2\pi/5)) \\ v_2 &= (0, \sin(2\pi/5), \sin(\pi/5), -\sin(\pi/5), -\sin(2\pi/5)) \end{aligned}$$

Example 2, icosahedral tiling of $\mathbb{R}^3$:

In $\mathbb{R}^6$, set $E = \mathrm{Im}\ \pi$ of dimension 3 with

$$\pi = \frac{1}{2\sqrt{5}} \begin{bmatrix} \sqrt{5} & 1 & -1 & -1 & 1 & 1 \\ 1 & \sqrt{5} & 1 & -1 & -1 & 1 \\ -1 & 1 & \sqrt{5} & 1 & -1 & 1 \\ -1 & -1 & 1 & \sqrt{5} & 1 & 1 \\ -1 & -1 & -1 & 1 & \sqrt{5} & 1 \\ 1 & 1 & 1 & 1 & 1 & \sqrt{5} \end{bmatrix}$$

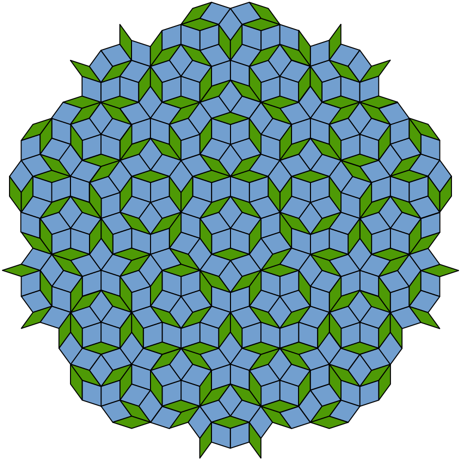

FIGURE: Third Penrose tiling

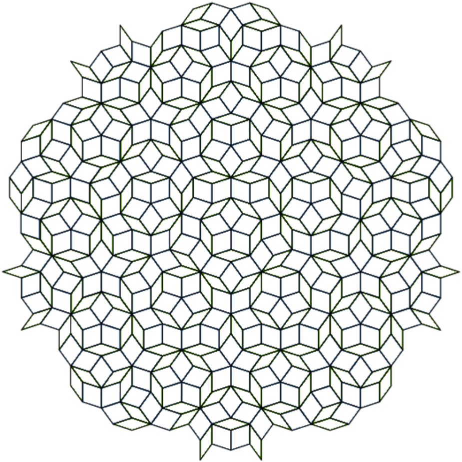

Figure: Third Penrose tiling

Theorem

Let $N > d \geq 1$. Under assumption **NDEG**, *for almost all $t \in E'$ non ambiguous, the following estimates holds :*

- $P^{2n}(x,x) \geq C_1[n(\log n)]^{-d/2}$,
- $P^n(x,y) \leq C_2 n^{-d/2}$,

for some constants $C_1, C_2 > 0$.

Corollary

Under the same assumptions, the simple random walk on the cut-and-project graph is recurrent for $d = 1, 2$ and transient otherwise.

Obviously, if $d = 1$, $P^{2n}(x,x) \sim_{n\to\infty} C_3 n^{-1/2}$, *this is the simple random walk on the integers.*

Theorem

Let $t = t_E + t_{E'} \in \mathbb{R}^N = E \oplus E'$ with $\dim E = d$. Then, under the assumption **NDEG**, *there exists an almost everywhere positive function ℓ on E' satisfying*

$$\lim_{r\to\infty} \frac{\operatorname{card}\Big(\mathbb{Z}^N \cap (t + B_{\mathbb{R}^d}(0,r) + K)\Big)}{\operatorname{Leb}(B_{\mathbb{R}^d}(0,r))} = \ell(t_{E'}), \quad a.e..$$

$\square$

- This theorem allows to compare the growth rate of balls and spheres in the cut-and-project graph embedded in $E \cong \mathbb{R}^d$ with the growth rate of balls and spheres in de $\mathbb{R}^d$.
- The Hof's theorem (1998) suppose in addition that $E' \cap \mathbb{Z}^N = \{0\}$. In this case, ℓ is constant and the convergence is uniform.

Isoperimetric inequalities

Proposition

Under assumption **NDEG**, *there exists a constant $K > 1$ such that for all $x \in \mathbb{G}^0$*

$$K^{-1}\ell^d \leq \mathrm{card} B_{\mathbb{G}}(x,\ell) \leq K\ell^d.$$

Proposition

Under assumption **NDEG**, *the graph $\mathbb{G}$ satisfies a d-dimensional isoperimetric inequality, i.e. there exists $K > 0$ such that*

$$\mathrm{card} B_{\mathbb{G}}(x,\ell) \leq K \mathrm{card} \partial B_{\mathbb{G}}(x,\ell)^{d/d-1}.$$

uniformly in $x \in \mathbb{G}^0$.

- Let $\mathbb{R}^N = E \oplus E'$ with $\dim E = d < N$ and $E' \perp E$.
- For simplicity, assume that $E' \cap \mathbb{Z}^N = \{0\}$ and $E \cap \mathbb{Z}^N = \{0\}$.
- Then, π and π', when restricted to $\mathbb{Z}^N$, are injective.
- Let $t \in E'$ be non ambiguous.
- Recall the definition of $\mathcal{T}_t$

$$\mathcal{T}_t = \left\{ x + D_I : x = \pi(\xi), \xi \in \mathbb{Z}^N : \pi'(\xi + K_I) \subset \pi'(K) + t, I \in M_d \right\}$$

- Let $\xi \in \mathbb{Z}^N \cap \mathscr{K}$, then the set of admissible neighbours is entirely defined through a local rule : $\pi'(\xi) \sim \pi'(\eta)$ iff

$$|\xi - \eta|_1 = 1 \text{ and } \exists \tilde{\eta}, I : \xi, \eta \in \tilde{\eta} + K_I \text{ and } \pi'(\tilde{\eta} + K_I) \subset \pi'(K) + t$$

- Forget the cut-and-project method.
- The local rule remains and defines a Markov chain Q on $\pi'(K) \subset E'$.
- The distribution of steps is uniform on admissible neighbours.
- Each admissible move in $\pi'(K)$ corresponds to a move along exactly one $\pm\varepsilon_i$.
- If $m(x)$ denotes the number of admissible neighbours of $x \in \pi'(K)$, then

$$\pi(dx) = \mathbf{1}_{\pi'(K)}(x) m(x) \lambda(dx)$$

  is a finite invariant measure for Q.

Summarizing : estimate the entropy of a Markov additive process with invariant probability measure and work a little bit to conclude.

Markov additive processes

Let $\mathbb{S}$ be a state space and Q a transition kernel on $\mathbb{S}$. We are given a family on probability measures $(\mu^{x,y})_{x,y\in\mathbb{S}}$ on $\mathbb{Z}^N$.

Definition

A Markov additive process is a Markov chain $((X_n, Z_n))_{n\geq 0}$ taking values in $\mathbb{S}\times\mathbb{Z}^N$ whose Markov operator is given by

$$Rf(x,z) = \sum_{y\in\mathbb{S}, z'\in\mathbb{Z}^N} Q(x,y)\mu^{x,y}(z')f(y, z+z'),$$

for $x\in\mathbb{S}$ et $z, z'\in\mathbb{Z}^N$ and $f\in\ell^\infty(\mathbb{S}\times\mathbb{Z}^N)$.

- $\mathcal{L}(X_n, Z_n - Z_{n-1} | (X_{n-1}, Z_{n-1}))$ only depends on X_{n-1} ;
- R commutes with the translations in $\mathbb{Z}^N$.

The simple case of entropy on groups

Let Γ be finitely generated group, $\mathcal{S}$ a symmetric finite set of generators (semigroup) and μ a probability supported by $\mathcal{S}$.

Let $g \in \Gamma$ and $\{X_n\}_{n\geq 1}$ be an *i.i.d.* sequence of Γ-valued random variable, the (right) random walk of law μ is defined as

$$Z_n = gX_1 \cdots X_n, \quad n \geq 1.$$

Definition (Shannon entropy)

The Shannon entropy of μ is given by

$$H(\mu) = -\sum_{g\in\Gamma} \mu(g) \log \mu(g).$$

Definition (Asymptotic entropy)

Denoting by μ^n the n-fold convolution of μ, the asymptotic entropy of the μ-random walk on Γ is given by

$$h = \lim_{n\to\infty} H(\mu^n) - H(\mu^{n-1}) = \lim_{n\to\infty} \frac{H(\mu^n)}{n}.$$

Shannon-McMillan-Breiman theorem

- For each $n \geq 1$, $X_n : \Gamma^{\mathbb{N}^*} \ni \omega \longrightarrow X_n(\omega) = \omega_n \in \Gamma$,
- let T be the Bernoulli shift on $(\Gamma^{\mathbb{N}^*}, \mu^{\mathbb{N}^*})$.

Then, for any $n, m \in \mathbb{N}^*$, using Γ-invariance

$$\mu^{n+m}(X_1 \cdots X_n X_{n+1} \cdots X_m) \geq \mu^n(X_1 \cdots X_n)\mu^m(X_{n+1} \cdots X_m)$$

Thus, $\omega \longrightarrow -\log \mu^n(Z_n(\omega))$ is a non negative subadditive cocycle and Kingmann subadditive theorem implies a SMB type result

$$\lim_{n\to\infty} -\frac{1}{n} \log \mu^n(Z_n) = \lim_{n\to\infty} \frac{1}{n} \mathbf{E}(-\log \mu^n(Z_n)) = h \quad \text{a.s..}$$

Fundamental inequality

Let d be the geodesic metric on $(\Gamma, \mathcal{S})$ and define (using Kingmann subadditive theorem again) the escape rate

$$\lim_{n\to\infty} \frac{1}{n} d(e, Z_n) = \lim_{n\to\infty} \frac{1}{n} \mathbf{E}[d(e, Z_n)] = \ell, \quad \text{a.s.}.$$

The fundamental inequality is an easy consequence of the SMB theorem :

$$h \leq \ell \cdot v, \quad \text{with} \quad v = \lim_{n\to\infty} \frac{1}{n} \log \operatorname{card} \ B_d(e, n).$$

For $\mathbb{Z}^N$, $v = 0$.

For Markov additive processes

Substitute the Bernoulli shift by the Markov shift

Adapt the definition of entropy and escape rate to the additive component

In fact : proved in Kaimanovich, Kiefer, Rubhstein (2004)

Deal with the fact the cut-and-project Markov shift is not ergodic !

Thank you !