Random walks on graphs induced by aperiodic tilings

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1 Introduction

2 Cut-and-project method

3 Recurrence/Transience

4 Asymptotic entropy
Seminal paper, Pólya, 1921

- $S = \{\pm \varepsilon_i : i = 1, \ldots, i = d\} \subset \mathbb{Z}^d$
- $\{\xi_k\}_{k \geq 1}$ a sequence of independent and identically distributed r.v. with common distribution $\mathbb{P}(\xi_1 = \pm \varepsilon_i) = 1/2d$
- $S_n = x + \sum_{k=1}^{n} \xi_k$

**Question**: What can we say about the probability

$$p = \mathbb{P}(S_n = x \text{ for infinitely many } n)?$$

**Answer (partial)**:

if $d = 1, 2$ $p = 1$ and $p = 0$ otherwise

**Remark**: By construction, the random walk is homogeneous in the sense that

$$\forall y \in \mathbb{Z}^d : \mathcal{L}(y + S_{n+1} | y + S_n) = \mathcal{L}(S_{n+1} | S_n)$$

**Motivation**: break the symmetries of the lattice
(Undirected) graphs and random walks

An undirected graph is a couple \((V, E)\) where

1. \(V\) is a countable set of nodes;
2. \(E\) is a set of unordered pairs \(\{x, y\}\) with \((x, y) \in V \times V\).

The degree, \(\text{deg } x\), of a nodes \(x \in V\) is \(\text{card} \{y \in V : \{x, y\} \in E\}\).

Generally, it is assumed that the graph \((V, E)\) is connected, locally finite and uniformly bounded which corresponds respectively to

1. for all \(x, y \in V\) there exists a path \((z_0 = x, z_1, \ldots, z_n = y)\) of nodes s.t. \(\{z_i, z_{i+1}\} \in E\) for \(i = 0, \ldots, n - 1\),
2. \(\text{deg } x < \infty\) for all \(x \in V\),
3. \(\sup_{x \in V} \text{deg } x < \infty\).
A random walk on a graph \((V, E)\) is a Markov chain \((M_n)_{n \geq 0}\) taking values in \(V\) whose transition kernel is adapted to the graph structure, \(i.e.\) for all \(x, y \in V\):

\[
P(x, y) > 0 \iff \{x, y\} \in E.
\]

A random walk on a graph is said to be simple if the transition kernel is given for all \(x, y \in V\) by

\[
P(x, y) = \begin{cases} 
\frac{1}{\deg(x)} & \text{if } \{x, y\} \in V, \\
0 & \text{otherwise.}
\end{cases}
\]
Example 1: homogeneous spaces

Let us denote by

- $\mathcal{G}(V)$ the group of permutations of $V$,
- $\text{Aut}(P) = \{g \in \mathcal{G}(V) : \forall x, y \in G^0, P(g \cdot x, g \cdot y) = P(x, y)\}$ the group of automorphism of $P$.

The Markov chain on $V$ with transition kernel $P$ is said **homogeneous** if the action of $\text{Aut}(P) \acts V$ is transitive.

Homogeneous MC $\iff$ RW on the group of automorphisms

Up to symmetries, they are processes with independent and stationnary increments
Example 2: triangulations of planar surface

Figure: Patch of a triangulation

Theorem (Dodziuk, 1984)

If for some $d \geq 0$

$$7 \leq \deg(x) \leq d$$

for all $x \in V$, then the simple random walk is transient.
Example 3: circle packing of the plane

**Theorem**

If \( \deg(x) \leq 6 \) for all but finitely many points \( x \), then the simple random walk is recurrent.
Example 3: Circle packing of the plane

**Figure:** Circle packing

**Theorem**

If $\deg(x) \leq 6$ for all but finitely many points $x$, then the simple random walk is recurrent.
Third Penrose tiling

Two prototiles:

- a fat rhombus (angles of measure $2\pi/5$ and $3\pi/5$),
- a thin rhombus (angles of measure $\pi/5$ and $4\pi/5$).

together with matching rules

\[ \downarrow \]

Aperiodic tiling of $\mathbb{R}^2$
Third Penrose tiling

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\[ \Downarrow \]
Aperiodic tiling of $\mathbb{R}^2$
Let

- $\mathbb{Z}^2 \subset \mathbb{R}^2$,
- $E$ a vector line with $E \cap \mathbb{Z}^2 = \{0\}$,
- $E'$ the orthogonal supplement of $E$,
- $K$ the unit cube of $\mathbb{Z}^2$

the strip $\mathcal{K}_t = K + E + t$, $t \in E'$.

Orthogonal projection of edges entirely contained in the strip

$\Downarrow$

Aperiodic tiling of $E$ with short and long segments
**Main goal**: tile a \(d\)-dimensional vector subspace \(E \subset \mathbb{R}^N\)

1. \(\mathbb{Z}^N \subset \mathbb{R}^N\), \(S = \{ \pm \varepsilon_i : 1 \leq i \leq N \}\),
2. \(\mathbb{R}^N = E \oplus E'\) avec \(\dim E = d\), \(\pi\) and \(\pi'\) the corresponding projectors,
3. \(K = \left\{ \sum_{i=1}^N \lambda_i \varepsilon_i : \lambda_i \in [0, 1], 1 \leq i \leq N \right\}\),
4. \(M_d = \left\{ I = \{i_1, \cdots, i_d\} \subset \{1, \cdots, N\} \right\}\),
5. \(K_I = \left\{ \sum_{\ell=1}^d \lambda_{i_\ell} \varepsilon_{i_\ell} : \lambda_{i_\ell} \in [0, 1], i_\ell \in I \right\}\),
6. \(K_{I^c}\) denotes the supplementary facet of \(K_I\) in the unit cube \(K\),
7. \(D_I = \pi(K_I)\) et \(D'_I = \pi'(K_{I^c})\),
8. **NDEG assumption**: \(d\)-dimensional facets of \(K \cong D_I\) & \((N - d)\)-dimensional facets of \(K \cong D'_I\).
Theorem (Oguey, Duneau and Katz)

For $t \in E'$, we set

$$\mathcal{T}_t = \left\{ x + D_I : x = \pi(\xi), \xi \in \mathbb{Z}^N : \pi'(\xi + K_I) \subset \pi'(K) + t, I \in M_d \right\}$$

Theorem (Oguey, Duneau, Katz, 1988)

The $\mathcal{T}_t$ is a tiling of $E \cong \mathbb{R}^d$ for all $t \in E'$ generic (or non ambiguous).

Remarks:

1. Meaning: there exists a unique $d$-dimensional faceted manifold entirely contained in the strip $\mathcal{H}_t$ (for any generic $t \in E'$);

2. Group of translations of the tiling: $E \cap \mathbb{Z}^N$. 
Examples

Example 1, the third Penrose tiling:
In $\mathbb{R}^5$, set $E = \text{Span}(v_1, v_2)$ with

\[
v_1 = (1, \cos(2\pi/5), -\cos(\pi/5), -\cos(\pi/5), \cos(2\pi/5))
v_2 = (0, \sin(2\pi/5), \sin(\pi/5), -\sin(\pi/5), -\sin(2\pi/5))
\]

Example 2, icosahedral tiling of $\mathbb{R}^3$:
In $\mathbb{R}^6$, set $E = \text{Im} \, \pi$ of dimension 3 with

\[
\pi = \frac{1}{2\sqrt{5}} \begin{bmatrix}
\sqrt{5} & 1 & -1 & -1 & 1 & 1 \\
1 & \sqrt{5} & 1 & -1 & -1 & 1 \\
-1 & 1 & \sqrt{5} & 1 & -1 & 1 \\
-1 & -1 & 1 & \sqrt{5} & 1 & 1 \\
-1 & -1 & -1 & 1 & \sqrt{5} & 1 \\
1 & 1 & 1 & 1 & 1 & \sqrt{5}
\end{bmatrix}
\]
**Figure**: Third Penrose tiling
Figure: Third Penrose tiling
Theorem

Let $N > d \geq 1$. Under assumption NDEG, for almost all $t \in E'$ non ambiguous, the following estimates holds:

- $P^{2n}(x, x) \geq C_1 [n(\log n)]^{-d/2}$,
- $P^n(x, y) \leq C_2 n^{-d/2}$,

for some constants $C_1, C_2 > 0$.

Corollary

Under the same assumptions, the simple random walk on the cut-and-project graph is recurrent for $d = 1, 2$ and transient otherwise.

Obviously, if $d = 1$, $P^{2n}(x, x) \sim_{n \to \infty} C_3 n^{-1/2}$, this is the simple random walk on the integers.
Theorem

Let $t = t_E + t_{E'} \in \mathbb{R}^N = E \oplus E'$ with $\dim E = d$. Then, under the assumption NDEG, there exists an almost everywhere positive function $\ell$ on $E'$ satisfying

$$\lim_{r \to \infty} \frac{\card\left(\mathbb{Z}^N \cap (t + B_{\mathbb{R}^d}(0, r) + K)\right)}{\text{Leb}(B_{\mathbb{R}^d}(0, r))} = \ell(t_{E'}), \text{ a.e.}$$

- This theorem allows to compare the growth rate of balls and spheres in the cut-and-project graph embedded in $E \cong \mathbb{R}^d$ with the growth rate of balls and spheres in de $\mathbb{R}^d$.
- The Hof’s theorem (1998) suppose in addition that $E' \cap \mathbb{Z}^N = \{0\}$. In this case, $\ell$ is constant and the convergence is uniform.
Isoperimetric inequalities

Proposition

Under assumption \textbf{NDEG}, there exists a constant $K > 1$ such that for all $x \in G^0$

$$K^{-1} \ell^d \leq \text{card} B_G(x, \ell) \leq K \ell^d.$$ 

Proposition

Under assumption \textbf{NDEG}, the graph $G$ satisfies a $d$-dimensional isoperimetric inequality, i.e. there exists $K > 0$ such that

$$\text{card} B_G(x, \ell) \leq K \text{card} \partial B_G(x, \ell)^{d/d-1}.$$

uniformly in $x \in G^0$.
Let \( \mathbb{R}^N = E \oplus E' \) with \( \dim E = d < N \) and \( E' \perp E \).

For simplicity, assume that \( E' \cap \mathbb{Z}^N = \{0\} \) and \( E \cap \mathbb{Z}^N = \{0\} \).

Then, \( \pi \) and \( \pi' \), when restricted to \( \mathbb{Z}^N \), are injective.

Let \( t \in E' \) be non ambiguous.

Recall the definition of \( \mathcal{T}_t \)

\[
\mathcal{T}_t = \left\{ x + D_I : x = \pi(\xi), \xi \in \mathbb{Z}^N : \pi'(\xi + K_I) \subset \pi'(K) + t, I \in M_d \right\}
\]

Let \( \xi \in \mathbb{Z}^N \cap \mathcal{K} \), then the set of admissible neighbours is entirely defined through a local rule: \( \pi'(\xi) \sim \pi'(\eta) \) iff

\[
|\xi - \eta|_1 = 1 \text{ and } \exists \tilde{\eta}, I : \xi, \eta \in \tilde{\eta} + K_I \text{ and } \pi'(\tilde{\eta} + K_I) \subset \pi'(K) + t
\]
● Forget the cut-and-project method.
● The local rule remains and defines a Markov chain \( Q \) on \( \pi'(K) \subset E' \).
● The distribution of steps is uniform on admissible neighbours.
● Each admissible move in \( \pi'(K) \) corresponds to a move along exactly one \( \pm \epsilon_i \).

If \( m(x) \) denotes the number of admissible neighbours of \( x \in \pi'(K) \), then

\[
\pi(dx) = 1_{\pi'(K)}(x)m(x)\lambda(dx)
\]

is a finite invariant measure for \( Q \).

**Summarizing**: estimate the entropy of a Markov additive process with invariant probability measure and work a little bit to conclude.
Markov additive processes

Let $\mathcal{S}$ be a state space and $Q$ a transition kernel on $\mathcal{S}$. We are given a family of probability measures $(\mu^{x,y})_{x,y \in \mathcal{S}}$ on $\mathbb{Z}^N$.

**Definition**

A Markov additive process is a Markov chain $((X_n, Z_n))_{n \geq 0}$ taking values in $\mathcal{S} \times \mathbb{Z}^N$ whose Markov operator is given by

$$Rf(x, z) = \sum_{y \in \mathcal{S}, z' \in \mathbb{Z}^N} Q(x, y) \mu^{x,y}(z') f(y, z + z'),$$

for $x \in \mathcal{S}$ and $z, z' \in \mathbb{Z}^N$ and $f \in \ell^\infty(\mathcal{S} \times \mathbb{Z}^N)$.

- $\mathcal{L}(X_n, Z_n - Z_{n-1}|(X_{n-1}, Z_{n-1}))$ only depends on $X_{n-1}$;
- $R$ commutes with the translations in $\mathbb{Z}^N$. 

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The simple case of entropy on groups

Let $\Gamma$ be finitely generated group, $S$ a symmetric finite set of generators (semigroup) and $\mu$ a probability supported by $S$.

Let $g \in \Gamma$ and $\{X_n\}_{n \geq 1}$ be an \textit{i.i.d.} sequence of $\Gamma$-valued random variable, the (right) random walk of law $\mu$ is defined as

$$Z_n = gX_1 \cdots X_n, \quad n \geq 1.$$
Definition (Shannon entropy)

The Shannon entropy of $\mu$ is given by

$$H(\mu) = -\sum_{g \in \Gamma} \mu(g) \log \mu(g).$$

Definition (Asymptotic entropy)

Denoting by $\mu^n$ the $n$-fold convolution of $\mu$, the asymptotic entropy of the $\mu$-random walk on $\Gamma$ is given by

$$h = \lim_{n \to \infty} H(\mu^n) - H(\mu^{n-1}) = \lim_{n \to \infty} \frac{H(\mu^n)}{n}.$$
For each $n \geq 1$, $X_n : \Gamma^\mathbb{N}^* \ni \omega \mapsto X_n(\omega) = \omega_n \in \Gamma$, let $T$ be the Bernoulli shift on $(\Gamma^\mathbb{N}^*, \mu^\mathbb{N})$.

Then, for any $n, m \in \mathbb{N}^*$, using $\Gamma$-invariance

$$\mu^{n+m}(X_1 \cdots X_n X_{n+1} \cdots X_m) \geq \mu^n(X_1 \cdots X_n) \mu^m(X_{n+1} \cdots X_m)$$

Thus, $\omega \mapsto -\log \mu^n(Z_n(\omega))$ is a non negative subadditive cocycle and Kingmann subadditive theorem implies a SMB type result

$$\lim_{n \to \infty} -\frac{1}{n} \log \mu^n(Z_n) = \lim_{n \to \infty} \frac{1}{n} \mathbb{E}(-\log \mu^n(Z_n)) = h \quad \text{a.s.}$$
Let $d$ be the geodesic metric on $(\Gamma, S)$ and define (using Kingmann subadditive theorem again) the escape rate

$$\lim_{n \to \infty} \frac{1}{n} d(e, Z_n) = \lim_{n \to \infty} \frac{1}{n} \mathbb{E}[d(e, Z_n)] = \ell, \quad \text{a.s.}$$

The fundamental inequality is an easy consequence of the SMB theorem:

$$h \leq \ell \cdot v,$$

with

$$v = \lim_{n \to \infty} \frac{1}{n} \log \text{card } B_d(e, n).$$

For $\mathbb{Z}^N$, $v = 0$. 
For Markov additive processes

Substitute the Bernoulli shift by the Markov shift

Adapt the definition of entropy and escape rate to the additive component


Deal with the fact the cut-and-project Markov shift is not ergodic!
Thank you!