PATTERN COMPLEXITY ON THE EDGE BETWEEN PERIODICITY AND APERIODICITY

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MORSE-HEDLUND THEOREM

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Bi-infinite word: a function $\mathbb{Z} \to \mathcal{A}$

Subword complexity P(n):

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P(2) = 3 

P(3) = 4 

P(4) = 5   P(5) = 5   P(6) = 5   P(7) = 5   . . .

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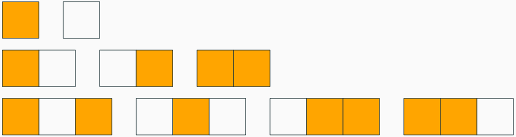

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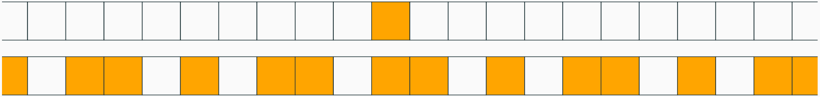
MORSE-HEDLUND THEOREM

Subword complexity $P(n)$:

# of distinct subwords of length n

$P(1) = 2$ $\quad$ $P(2) = 3$ $\quad$ $P(3) = 4$ $\quad$ $5 = P(4) = P(5) = P(6) = \ldots$



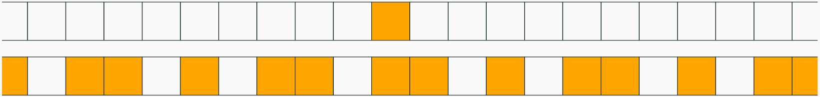
$P(n) = n+1$

MORSE-HEDLUND THEOREM

Theorem [Morse, Hedlund 1938]:

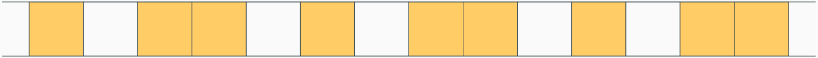
$$\text{a word is non-periodic} \quad \Leftrightarrow \quad \forall n : P(n) \geq n+1$$

P(1) = 2 P(2) = 3 P(3) = 4 5 = P(4) = P(5) = P(6) = . . .



P(n) = n+1

Theorem: $\exists n : P(n) \leq n \quad \Leftrightarrow \quad$ periodic



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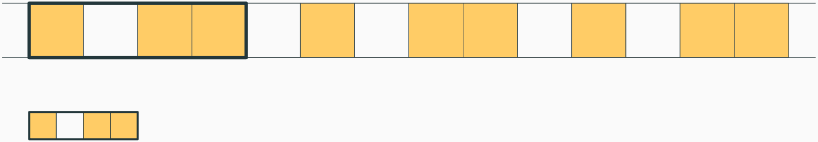

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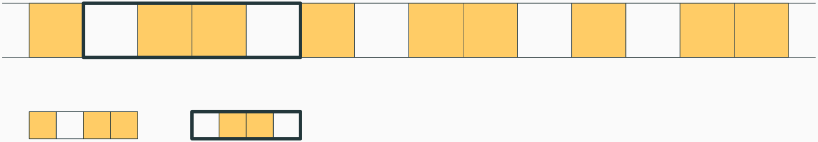

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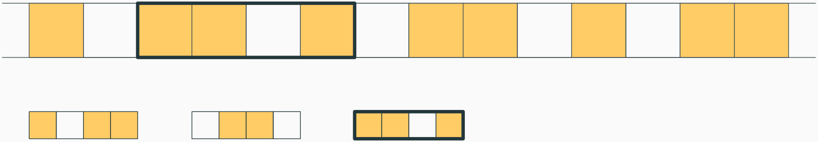

MORSE-HEDLUND THEOREM – PROOF

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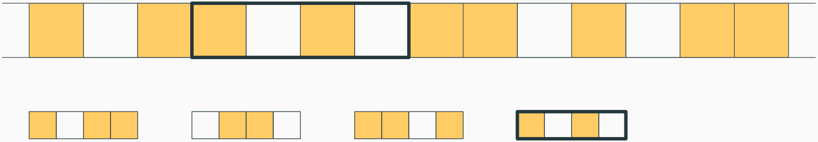

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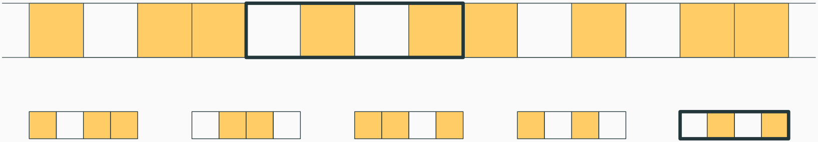

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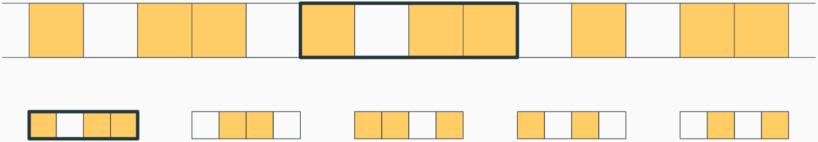

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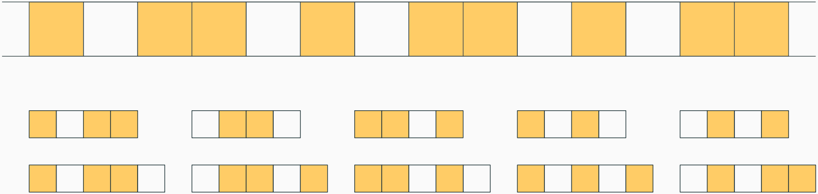

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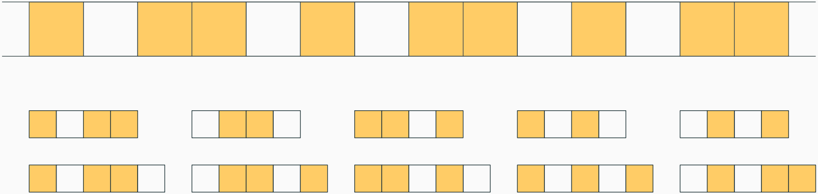

Proof.

- Let n be minimal, $P(1) \leq 1$ is ok

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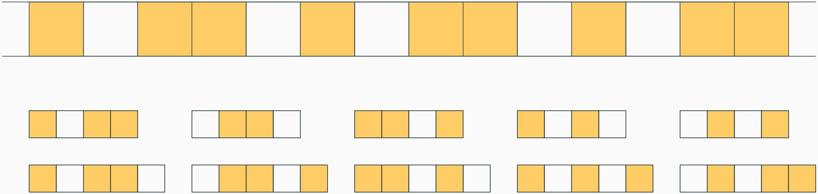
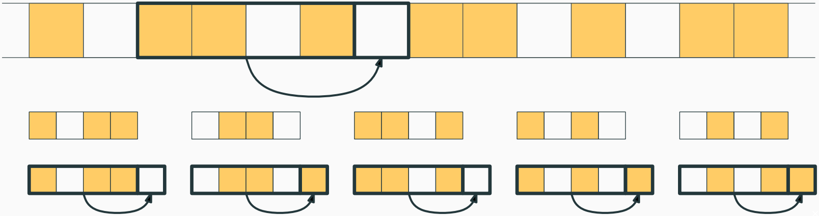


Proof.

- Let n be minimal, $P(1) \leq 1$ is ok
- $P(n-1) = P(n) = n$ because $n - 1 < P(n-1) \leq P(n) \leq n$

Theorem: $\exists n : P(n) \leq n \quad \Leftrightarrow \quad$ periodic



Proof.

- Let n be minimal, $P(1) \leq 1$ is ok
- $P(n-1) = P(n) = n$ because $n - 1 < P(n-1) \leq P(n) \leq n$
- determinism

STURMIAN WORDS

Morse-Hedlund thm: aperiodic $\Leftrightarrow$ $\forall n : P(n) \geq n+1$

Sturmian words: $\forall n : P(n) = n+1$

Fibonacci word:

- 0
- 01
- 010
- 01001
- 01001010
- 0100101001001
- ...

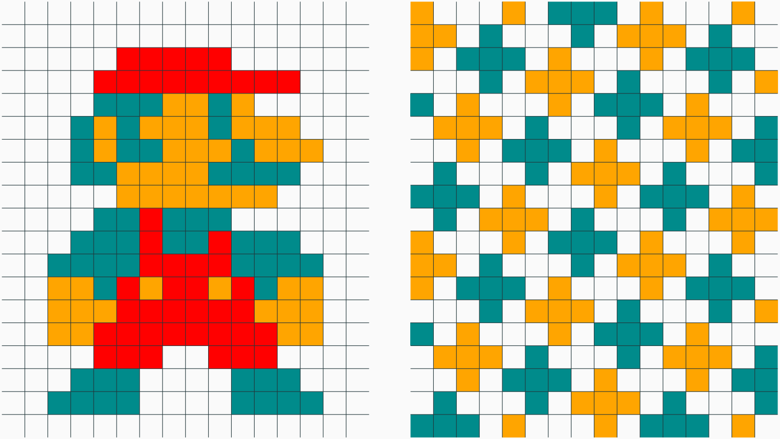
NIVAT'S CONJECTURE

Nivat's Conjecture

- two-dimensional generalization
- by Maurice Nivat, ICALP 1997

Let $\mathcal{A}$ be a finite set.

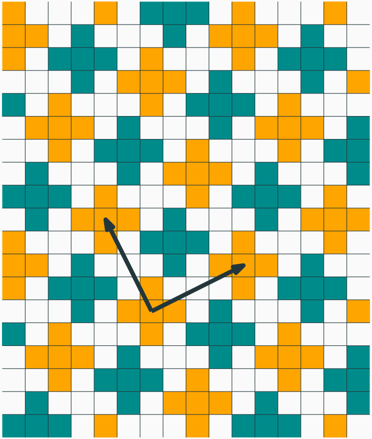
A d-dimensional configuration: a function $\mathbb{Z}^{d} \to \mathcal{A}$



Periodic:

$\exists\, \vec{v} \in \mathbb{Z}^2$ non-zero vector
such that translation by $\vec{v}$
preserves the image

periodic $\longrightarrow$



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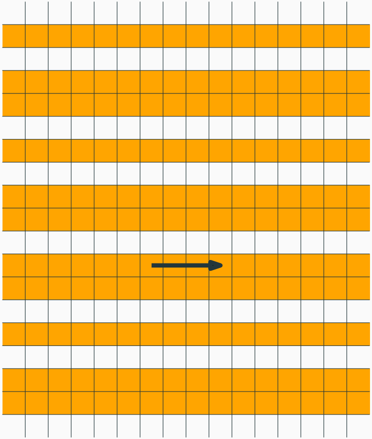
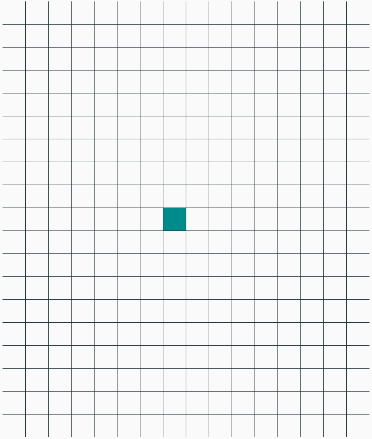


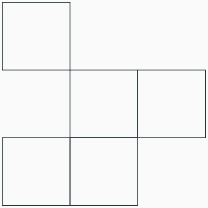
Periodic:

$\exists\, \vec{v} \in \mathbb{Z}^2$ non-zero vector such that translation by $\vec{v}$ preserves the image

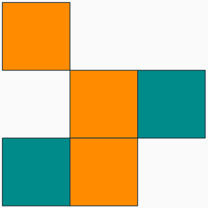
non-periodic $\longrightarrow$



A shape: $D \subset \mathbb{Z}^d$ finite



A D**-pattern:** $p \in \mathcal{A}^D$

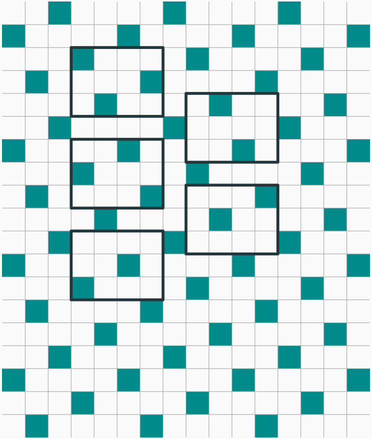


Pattern complexity $P(D)$**:** # of distinct patterns of shape D

In 2D,
rectangle complexity $P(m, n)$**:**

# of distinct $m \times n$ block patterns

$P(4,3) = 5$



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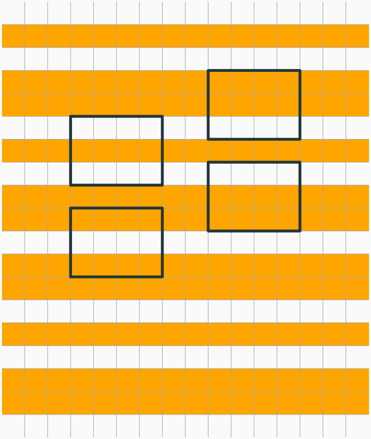


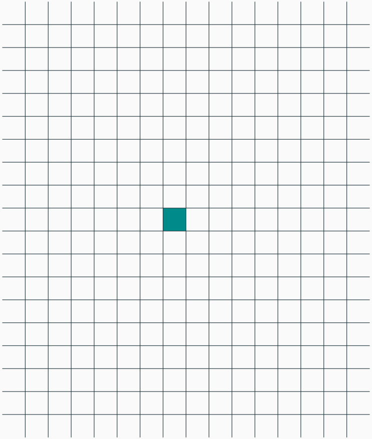
P(4,3) = 4
P(m,n) = n+1

In 2D,

rectangle complexity $P(m, n)$:

# of distinct $m \times n$ block patterns

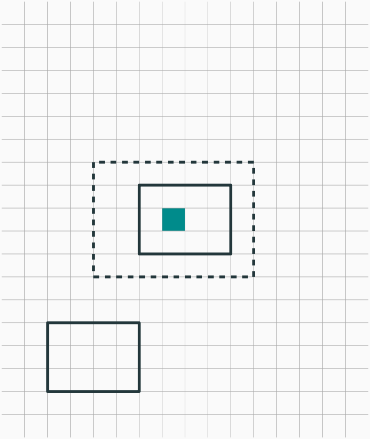
$P(4,3) = ?$



In 2D,
rectangle complexity $P(m, n)$:

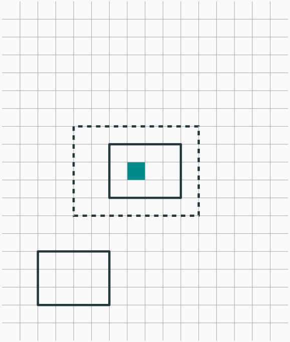
# of distinct $m \times n$ block patterns

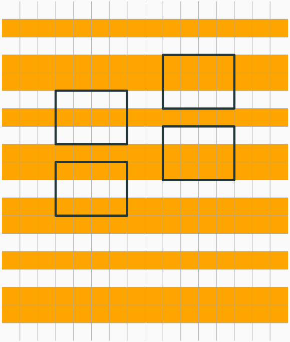
$P(4,3) = 13$
$P(m,n) = mn+1$



Nivat's Conjecture [Nivat, 1997]:

$$\text{a configuration is non-periodic} \quad \Rightarrow \quad \forall m, n\colon P(m,n) \geq mn + 1$$





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Known results [Van Cyr, Bryna Kra 2013]:

- a configuration is non-periodic $\Rightarrow$ $\forall m, n : P(m, n) > mn/2$
- a configuration is non-periodic $\Rightarrow$ $\forall m : P(m, 3) \geq 3m + 1$

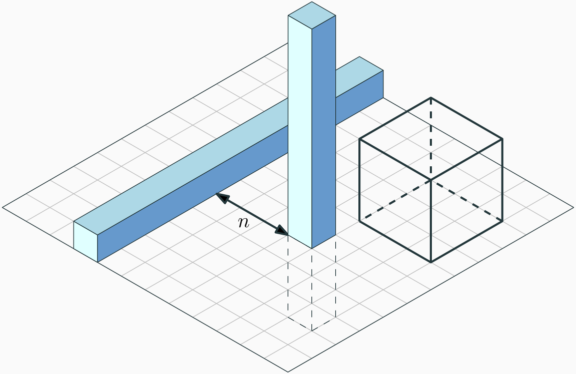
Nivat's Conjecture [Nivat, 1997]:

$$\text{a configuration is non-periodic} \quad \Rightarrow \quad \forall m, n : P(m, n) \geq mn + 1$$

Theorem [K., S. 2015]:

$$\text{a configuration is non-periodic}$$
$$\Downarrow$$
$$\text{for all but finitely many pairs m,n: } P(m, n) \geq mn + 1$$

Note: Generalization to 3D is false



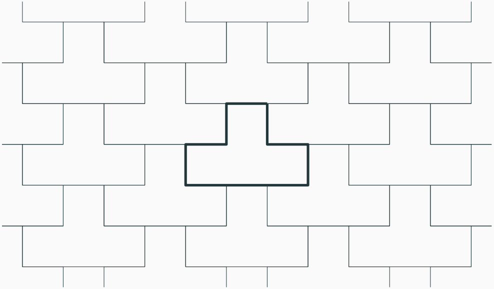
non-periodic but $P(n, n, n) = 2n^2 + 1 \ll n^3$

PERIODIC TILING PROBLEM

Tile: $T \subset \mathbb{Z}^d$ finite.

Conjecture [Lagarias, Wang 1996]:

$$\text{A tiling exists} \quad \Rightarrow \quad \text{a periodic tiling exists.}$$



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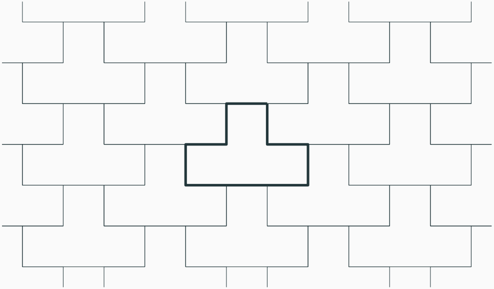

- **True** if prime size
- **True** if $d = 2$ [Bhattacharya 2016]

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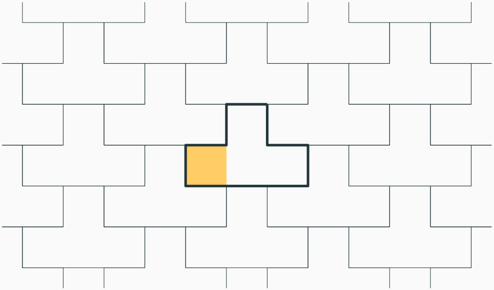

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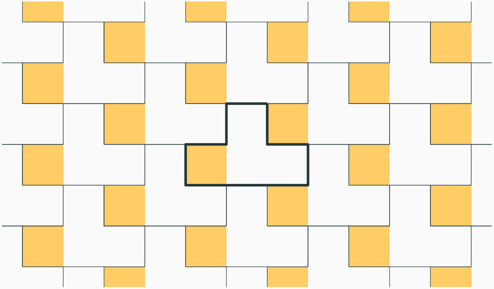

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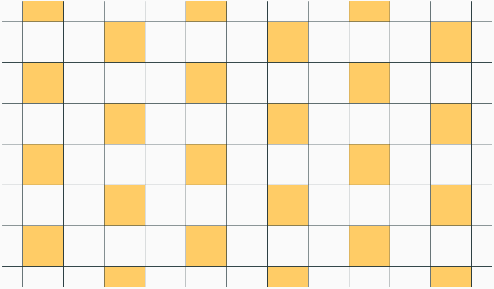

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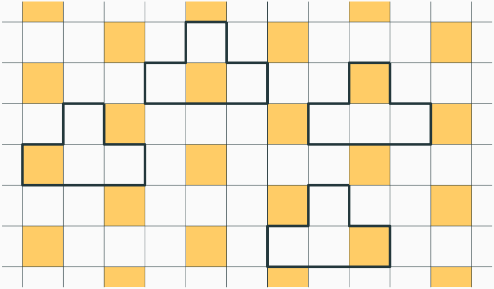
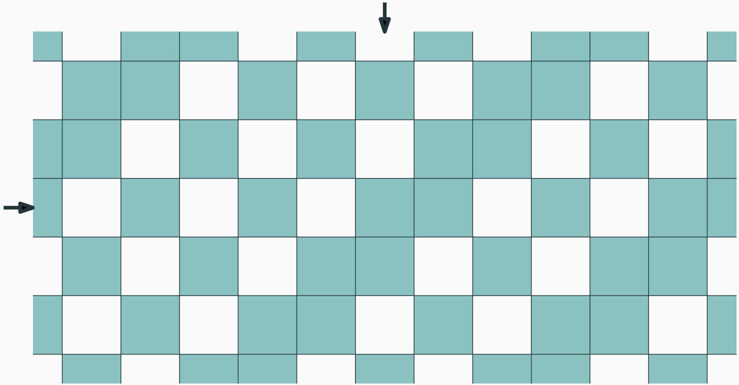


Low complexity condition: There exists a shape D s.t.

$$\boxed{P(D) \leq |D|}$$

- D is a rectangle $m \times n$ for Nivat's conjecture
- D is the tile for periodic tiling problem

THE POLYNOMIAL METHOD



FORMAL POWER SERIES

1	1	0	1	0	1	0	1	1	0	1
1	0	1	0	1	0	1	1	0	1	0
0	1	0	1	0	1	1	0	1	0	1
1	0	1	0	1	1	0	1	0	1	1
0	1	0	1	1	0	1	0	1	1	0

FORMAL POWER SERIES

$x^{-5}y^2$	$x^{-4}y^2$	0	$x^{-2}y^2$	0	y^2	0	x^2y^2	x^3y^2	0	x^5y^2
$x^{-5}y$	0	$x^{-3}y$	0	$x^{-1}y$	0	xy	x^2y	0	x^4y	0
0	x^{-4}	0	x^{-2}	0	1	x	0	x^3	0	x^5
$x^{-5}y^{-1}$	0	$x^{-3}y^{-1}$	0	$x^{-1}y^{-1}$	y^{-1}	0	x^2y^{-1}	0	x^4y^{-1}	x^5y^{-1}
0	$x^{-4}y^{-2}$	0	$x^{-2}y^{-2}$	$x^{-1}y^{-2}$	0	xy^{-2}	0	x^3y^{-2}	x^4y^{-2}	0

At position (i, j) : $c_{ij}x^iy^j$

$$x^{-5}y^2 + x^{-4}y^2 + 0 + x^{-2}y^2 + 0 + y^2 + 0 + x^2y^2 + x^3y^2 + 0 + x^5y^2$$

$$x^{-5}y + 0 + x^{-3}y + 0 + x^{-1}y + 0 + xy + x^2y + 0 + x^4y + 0$$

$$0 + x^{-4} + 0 + x^{-2} + 0 + 1 + x + 0 + x^3 + 0 + x^5$$

$$x^{-5}y^{-1} + 0 + x^{-3}y^{-1} + 0 + x^{-1}y^{-1} + y^{-1} + 0 + x^2y^{-1} + 0 + x^4y^{-1} + x^5y^{-1}$$

$$0 + x^{-4}y^{-2} + 0 + x^{-2}y^{-2} + x^{-1}y^{-2} + 0 + xy^{-2} + 0 + x^3y^{-2} + x^4y^{-2} + 0$$

$$\sum_{(i,j)\in\mathbb{Z}^2} c_{ij}x^iy^j$$

Configuration: formal power series over $\mathbb{C}$

$$c(x, y) = \sum_{(i,j)\in\mathbb{Z}^2} c_{ij} x^i y^j$$

integral: coefficients from $\mathbb{Z}$

finitary: finitely many distinct coefficients

Configuration: formal power series $c \in \mathbb{C}[[X^{\pm 1}]]$

$$c(X) = \sum_{\mathbf{v} \in \mathbb{Z}^d} c_{\mathbf{v}} X^{\mathbf{v}}$$

SIMPLIFIED NOTATION

integral: coefficients from $\mathbb{Z}$

finitary: finitely many distinct coefficients

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$$c(X) = \sum_{\mathbf{v} \in \mathbb{Z}^d} c_{\mathbf{v}} X^{\mathbf{v}}$$

integral: coefficients from $\mathbb{Z}$

finitary: finitely many distinct coefficients

$$\boxed{\text{symbolic configuration} \quad \longleftrightarrow \quad \text{finitary integral configuration}}$$

Configuration: formal power series over $\mathbb{C}$

$$c(x, y) = \sum_{(i,j)\in\mathbb{Z}^2} c_{ij} x^i y^j$$

Question: What happens if $c(x, y)$ is multiplied by $x^a y^b$?

Configuration: formal power series over $\mathbb{C}$

$$c(X) = \sum_{\mathbf{v} \in \mathbb{Z}^d} c_{\mathbf{v}} X^{\mathbf{v}}$$

SIMPLIFIED NOTATION

Question: What happens if c is multiplied by $X^{\mathbf{u}}$?

Answer: The configuration translates by the vector $\mathbf{u}$!

Configuration: formal power series over $\mathbb{C}$

$$c(X) = \sum_{\mathbf{v} \in \mathbb{Z}^d} c_{\mathbf{v}} X^{\mathbf{v}}$$

Question: What happens if c is multiplied by $X^{\mathbf{u}}$?

Answer: The configuration translates by the vector $\mathbf{u}$!

Observe: Configuration is periodic iff $\exists\, \mathbf{u} \neq 0$:

$$\begin{aligned} & X^{\mathbf{u}} c = c \\ \Leftrightarrow \quad & (X^{\mathbf{u}} - 1)c = 0 \end{aligned}$$

ANNIHILATING LAURENT POLYNOMIALS

$$\boxed{\operatorname{Ann}(c) = \left\{ f(x,y) \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}] \;\middle|\; f(x,y)c(x,y) = 0 \right\}}$$

$$\boxed{\mathrm{Ann}(c) = \{\, f \in \mathbb{C}[X^{\pm 1}] \mid fc = 0 \,\}}$$

SIMPLIFIED NOTATION

- **Annihilator ideal:** $\mathrm{Ann}(c)$
- **Annihilator polynomial:** $f \in \mathbb{C}[X]$ such that $fc = 0$
- **Observe:** c is periodic iff for some non-zero $\mathbf{u} \in \mathbb{Z}^d$

$$X^{\mathbf{u}} - 1 \in \mathrm{Ann}(c)$$

Lemma: If exists a shape D such that $P(D) \leq |D|$, then $\mathrm{Ann}(c) \neq \{0\}$.

Proof. Linear algebra.

Theorem: Let c be a finitary integral configuration with $\mathrm{Ann}(c) \neq \{0\}$. Then $\exists \mathbf{u}_1, \ldots, \mathbf{u}_n \in \mathbb{Z}^d$ non-zero such that

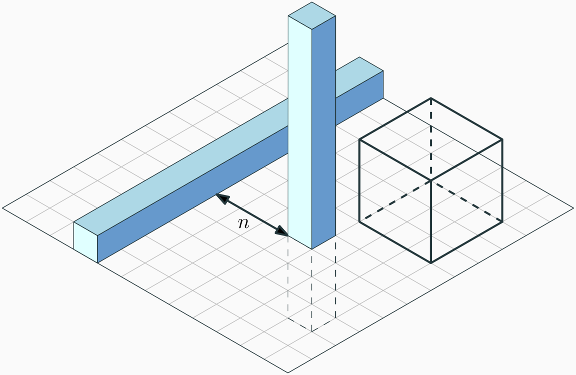
$$(X^{\mathbf{u}_1} - 1) \cdots (X^{\mathbf{u}_n} - 1) \in \mathrm{Ann}(c).$$

Proof. Hilbert's nullstellensatz.

Decomposition theorem: Let c be a finitary integral configuration with $\mathrm{Ann}(c) \neq \{0\}$. Then $\exists c_1, \ldots, c_n \in \mathbb{C}[X^{\pm 1}]$ periodic such that $c = c_1 + \cdots + c_n$.

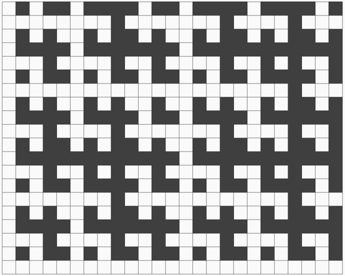
Proof. Corollary of the former.

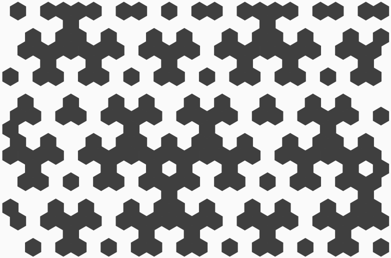
$\mathrm{P}(3,3,3) = 19 \leq 3 \cdot 3 \cdot 3$



$$c = c_3 - c_1 - c_2$$

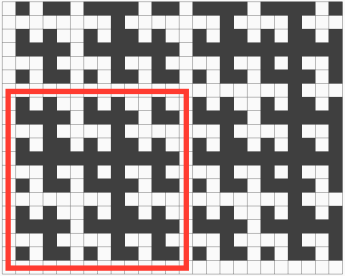
$$c_1(i,j) = \lfloor i\alpha \rfloor, \qquad c_2(i,j) = \lfloor j\alpha \rfloor, \qquad c_3(i,j) = \lfloor (i+j)\alpha \rfloor$$

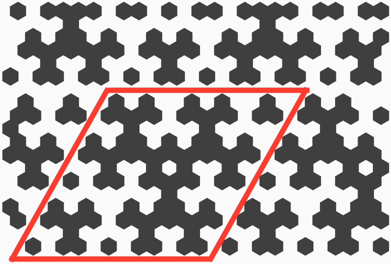




$$c = c_3 - c_1 - c_2$$

$$c_1(i, j) = \lfloor i\alpha \rfloor, \qquad c_2(i, j) = \lfloor j\alpha \rfloor, \qquad c_3(i, j) = \lfloor (i + j)\alpha \rfloor$$





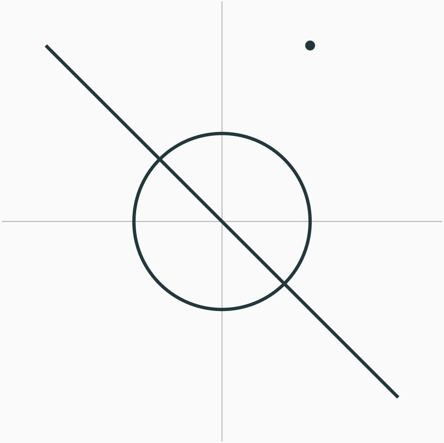
THE ANNIHILATOR IDEAL

Let $A \leq \mathbb{C}[X]$ be an ideal.

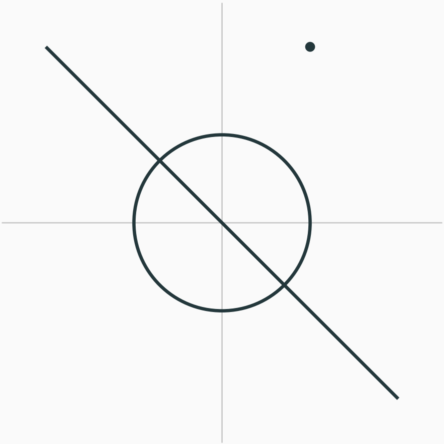
- $\sqrt{A} = \{\, f \in \mathbb{C}[X] \mid \exists n \colon f^n \in A \,\}$
- A is a radical ideal if $A = \sqrt{A}$
- A is a prime ideal if $ab \in A \Rightarrow a \in A \vee b \in A$

Theorem (Minimal decomposition): Let R be an algebraically closed field. Every radical ideal $A \leq R[X]$ can be uniquely written as a finite intersection of prime ideals $A = P_1 \cap \cdots \cap P_k$ where $P_i \not\subset P_j$ for $i \neq j$.

PRIME DECOMPOSITION OF RADICALS



- $\langle x^2 + y^2 - 1 \rangle$
- $\langle x^2 + 2xy + y^2 \rangle$
- $\langle x - 1, y - 2 \rangle$
- $\langle x^2 + y^2 - 1, x + y \rangle$
- $\langle (x^2 + y^2 - 1)(x - 1),$
  $(x^2 + y^2 - 1)(y - 2) \rangle$



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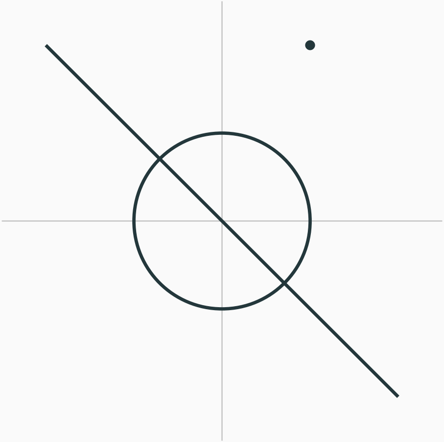
- $\langle x - 1, y - 2 \rangle$
- $\langle x^2 + y^2 - 1, x + y \rangle$
- $\langle (x^2 + y^2 - 1)(x - 1),$
  $(x^2 + y^2 - 1)(y - 2) \rangle$

Theorem. Let $A \leq \mathbb{C}[X]$ be a radical ideal. Then A can be uniquely written as a finite intersection of prime ideals $P_1 \cap \cdots \cap P_k$ such that $P_i \not\subset P_j$ for $i \neq j$.



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- $\langle x^2 + y^2 - 1, x + y \rangle$
- $\langle (x^2 + y^2 - 1)(x - 1),$
  $(x^2 + y^2 - 1)(y - 2) \rangle$

Lemma. Non-zero prime ideals of $\mathbb{C}[x, y]$ are:

- $\langle \varphi \rangle$ for an irreducible polynomial $\phi \in \mathbb{C}[x, y]$
- maximal ideals $\langle x - \alpha, y - \beta \rangle$

From now on, $\mathbf{d} = \mathbf{2}$.

Theorem: Let c be a 2D configuration. Then Ann(c) is a radical ideal.

Corollary: Let c be a 2D configuration. Then

$$\mathrm{Ann}(c) = \varphi_1 \cdots \varphi_k H$$

where $\varphi_i = X^{\mathbf{u}_i} - \omega_i$ with $\mathbf{u}_i \in \mathbb{Z}^2$, ω_i a root of 1 and H an intersection of maximal ideals.

Proof.

- $(X^{\mathbf{u}_1} - 1) \cdots (X^{\mathbf{u}_n} - 1) \in \mathrm{Ann}(c)$
- $A \cap B = AB$ for comaximal ideals

Corollary: Let c be a 2D configuration. Then

$$\mathrm{Ann}(c) = \phi_1 \cdots \phi_m H$$

where ϕ_i are line polynomials in distinct directions and H an intersection of maximal ideals.

Corollary: Let c be a 2D configuration. Then

$$\mathrm{Ann}(c) = \phi_1 \cdots \phi_m H$$

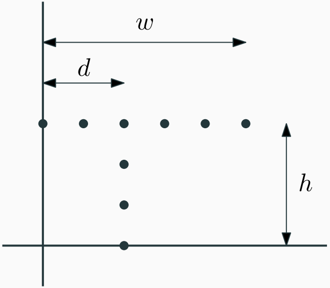
where ϕ_i are line polynomials in distinct directions and H an intersection of maximal ideals.

Definition: $\mathrm{ord}(c) = m$

Lemma:

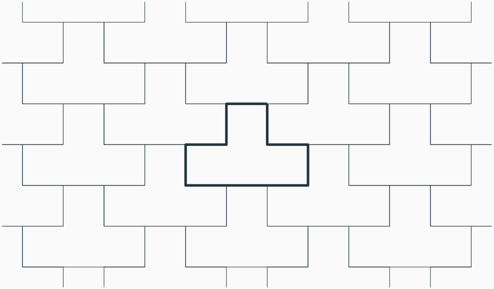
- $\mathrm{ord}(c) = 0$ iff c is doubly periodic
- $\mathrm{ord}(c) = 1$ iff c is one-way periodic
- $\mathrm{ord}(c) \geq 2$ iff c is non-periodic

T-shape:



Lemma: If $P(D) \leq |D|$ for a T-shape D, then c is periodic.

Lemma: Periodic tiling problem holds for $|T| = 4$.



Questions?